# Second-Order Risk of Alternative Risk Parity Strategies

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Abstract: The concept of second-order risk operationalizes the estimation risk induced by model uncertainty in portfolio construction. We study its contribution to the realized volatility of recently developed alternative risk parity strategies that invest in an uncorrelated decomposition of the asset universe. For each strategy, we derive closed-form solutions for the second-order risk, subsequently illustrated in empirical analysis based on real market data. Our results suggest a relation between the contribution of second-order risk and the sensitivity of a portfolio to single eigenvectors of the covariance matrix of assets' returns. Among the strategies considered, we find the principal risk parity strategy that invests equally in each eigenvector underlying the variance–covariance matrix to be immune to second-order risk. For the other strategies, second-order risk can be partially mitigated by means of statistical methods. In particular, we provide evidence for the eigenvalue adjustment being the most effective method for correcting the second-order risk bias.

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# Short-run Risk, Business Cycle, and the Value Premium *

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**Abstract**

We jointly explain the variations of the equity and value premium in a model with both short-run (SRR) and long-run (LRR) consumption risk. In our preliminary empirical analysis, we find that SRR varies with the business cycle and it has a substantial predictive power for market excess returns and the value premium—both in-sample and out-of-sample. The LRR component also differs significantly from zero, and value stocks have a larger exposure to both LRR and SRR than growth stocks. To explain these patterns in asset returns, we propose an extended and analytically tractable LRR model.

JEL classification: C32, G12, E44

Key Words: Long-run and short-run consumption risk, value premium, business cycle, portfolio selection, stochastic covariance.

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The value premium refers to the phenomenon that stocks with lower price-to-fundamentals ratios will generate excess returns over those with high ratios. This differential in returns constitutes a puzzle because the return differential cannot be accounted for by CAPM, as documented in Fama and French (1992). A large body of research has tried to reconcile data with theory. A prominent example is the long-run risk (henceforth LRR) model proposed in Bansal and Yaron (2004), which can explain the magnitude of both the equity and value premium.[1] Through a constant leverage parameter, value stocks load more on the LRR component than growth stocks. Therefore, investors require compensation for bearing more LRR, thus generating the value premium.[2] However, the assumption of a constant leverage parameter is a serious drawback of the LRR approach because it fails to explain the variation of equity and value premiums over business cycles. Moreover, the model's design is complicated by the commonly found evidence that the equity premium is pro-cyclical while the value premium is counter-cyclical.[3]

Our paper makes two contributions. The first contribution is to construct non-parametric measures of short-run risk (SRR) to formally study the covariation with the transient consumption growth as indicators of the business cycle. These measures are motivated by the specification of consumption and cash flow dynamics in the LRR framework, although they do not depend on the equilibrium solutions. Hence, the empirical SRR measures corresponds exactly to a theoretical counterpart in the model dynamics using the transient shocks. We define the SRR in dividends as the short-run covariance with the consumption growth. Similarly, we define the SRR in consumption growth as its short-run variance.

The SRRs fluctuate substantially with business cycles and can even switch sign: the SRR in value stocks appears counter-cyclical, while the SRR in growth stocks seems pro-cyclical. By running predictive regressions of future returns on the estimated SRRs, we find that the SRRs in consumption, growth stocks, and value stocks explain 17.6% of the variations in the future one-year market excess returns, and 11.5% of the variations in the future three-year return dif-

[1] Other plausible explanations include the over-optimism of extrapolative investors (Bondt and Thaler, 1985), the growth options inherent in growth stocks (Zhang, 2005), cash flow duration (Lettau and Wachter, 2007) and disaster risk exposure (Tsai and Wachter, 2015).

[2] See e.g., Bansal, Dittmar, and Lundblad (2005a), Bansal, Kiku, Shaliastovich, and Yaron (2014), Parker and Julliard (2005), and Hansen, Heaton, and Li (2008).

[3] The counter-cyclicality of the value premium is provided, for example, in Lettau and Ludvigson (2001b), Petkova and Zhang (2005). Meanwhile, Koijen, Lustig, and Nieuwerburgh (2017) also find that during economic downturns, prices and dividend payouts of value stocks plunge, while those of growth stocks are less affected.

ferentials in value and growth stocks (value-minus-growth returns henceforth). In particular, the SRR in consumption negatively predicts future market returns but positively predicts future value-minus-growth returns, which echoes the evidences that the value premium is counter-cyclical. The regression coefficients on the SRRs are statistically significant. These results are consistent with the interpretation that the predictive power of SRRs stems from the comovements of both SRRs and market returns with the business cycle.

The predictive power of SRR on future returns is not only statistically but also economically significant. Notwithstanding the criticism in Welch and Goyal (2008) that most predictive regressions cannot beat historical average out-of-sample (OOS), the predictive power of SRRs remains strong OOS. Based on predictive regressions, we construct a market-timing strategy that adjusts the positions on the market portfolio and the risk-free asset once every year. This strategy doubles the Sharpe ratio of the market excess returns.

A byproduct of LRR model motivated SRR measures is that we can jointly study SRR and LRR. In a generalized method of moments (GMM) estimation, the null hypothesis of no LRR is rejected at 99.9% significance level. We find the LRR is persistent but large in magnitude relative to consumption growth. The value stocks not only have more exposure to LRR than growth stocks, consistent with Bansal, Dittmar, and Lundblad (2005a) and Parker and Julliard (2005), but they also have larger SRRs than growth stocks.

The second contribution is to extend the LRR model to account for the relationship between SRRs and cyclical variation in the equity and value premium. The model can be solved in quasi-closed form up to Riccati equations and is hence analytically tractable. We model an economy explicitly with a market portfolio and portfolios of growth and value stocks. Guided by Santos and Veronesi (2006, 2010) and Menzly, Santos, and Veronesi (2004), the market portfolio and the cross-section of stocks should be studied jointly to provide a consistent explanation for the stylized facts of asset returns. To account for the time-variation of the SRRs across business cycles, we model stochastic covariances explicitly as state variables.[4] The resulting dynamic covariance structure in

[4]We model stochastic covariances using a Wishart process. The theoretical foundations of Wishart processes are laid out in Bru (1991) and introduced to finance by Gourieroux and Sufana (2003). Buraschi, Cieslak, and Trojani (2008) subsequently use a Wishart covariance process to study the term structure of interest rates. For derivative pricing, we refer to Gourieroux and Sufana (2004) and Gruber et al. (2015); and for portfolio choice, see Buraschi et al. (2010). More recently, Cieslak and Povala (2016a) exploit the properties of the Wishart process to reflect a

the cross section of assets is necessary to resolve the negative correlation between the market risk premium and SRR in consumption, which would otherwise be positive in a univariate volatility setting.

Our extended LRR model retains its analytical tractability, which allows us to calibrate the model to match the dynamics in the growth of consumption and dividends, the time-varying SRRs, and asset pricing patterns, such as the equity premium, the value premium, and the price-dividend ratios. Under the calibrations, the model matches market data reasonably well. In particular, the model replicates the predictive power of SRRs on the future market returns and value-minus-growth returns.

We also perform a series of robustness checks on our results. For the empirical studies, the results adopting alternative measures of cash flows by accounting for repurchases remain qualitatively the same. For the model, Pohl, Schmedders, and Wilms (2018) demonstrated that the potential errors induced by the log-linear approximation could be considerable. Consequently, we solve the model via the projection method and find that the errors in our case do not materially affect the model's results. Furthermore, we present empirical results where the SRRs are constructed from the monthly industrial production index instead of monthly consumption. Nonetheless, the industrial production index cannot explain future excess returns. Our regression exercise reveals a fundamental difference in the nature of consumption and industrial production data, rather than measurement errors.

Our paper shares many features with Bansal and Yaron (2004), albeit with some differences. First, Bansal and Yaron (2004) specify the growth rate dynamics of the monthly aggregated consumption, but we specify the growth rate dynamics of the annually aggregated consumption. Thus, in our paper, the growth rates of annually aggregated consumption can be represented by integrals, while in Bansal and Yaron (2004) the growth rates of annually aggregated consumption are approximated by the weighted-average of growth rates of monthly consumption. Our approach enables us to estimate SRR nonparametrically by realized variances or covariances, which is not done in Bansal and Yaron (2004). Second, we analyze the predictability of SRRs for market excess returns and value-minus-growth returns. Finally, we identify a persistent and large LRR component in a GMM

time-varying correlation between short-rate expectations and term premia, which is a difficult feature to achieve with traditional exponential affine models.

exercise. In comparison, Bansal and Yaron (2004) documents that the long-run risk component is persistent but small in the magnitude.

Bansal et al. (2005b) finds that the conditional volatility of consumption negatively predicts asset valuation ratios, which highlights the role of fluctuating economic uncertainty in asset markets. Consistent with these result, we also find that the SRR in consumption predicts future equity premia with a negative sign. Our paper additionally finds that the SRRs in value and growth stocks relate to asset valuations, and that the valuation of the value-minus-growth portfolio is also exposed to the fluctuating economic uncertainty. Furthermore, we formally extend and derive an analytically tractable long-run risk model, such that the SRRs correspond exactly to the model as measures of fluctuating economic uncertainty.

Boguth and Kuehn (2013) find that the volatility of the dividend growth of value stocks is more sensitive to the volatility of consumption growth than that of growth stocks, which underscores the importance of transient consumption innovations on the value premium. Our paper differs in several aspects. First, instead of studying the loadings on consumption growth volatility, this paper proposes the time-varying SRR. Second, Boguth and Kuehn (2013) conduct contemporaneous Fama-Macbeth regressions on consumption volatility. While they were able to sort firms according to their exposure to consumption volatility, the variation in the value premium per se is not studied. Instead, our goal is to explain the variation in the equity and value premiums, for which we perform the predictive regressions using SRRs.

This paper is closely related to Li and Zhang (2017), who jointly study the cross-sectional returns with LRR and SRR components in cash flows. Our paper differs in several areas. First, Li and Zhang (2017) define the SRR component as the regression coefficient of the biannual moving average of consumption growth on the dividend growth, and the LRR component as the covariation between dividend growth and the moving average of consumption growth in the last decades. Meanwhile, our definitions of SRR and LRR is different and consistent with the LRR model. Second, Li and Zhang (2017) attributes the value premium to exposures on LRR, and the momentum returns to exposures on SRR. In contrast, our paper focuses on the implications of SRR to the variation in the market equity premium and value premium. Third, Li and Zhang (2017) simulates a large cross section of firms and form portfolios on those firms. In contrast, we model growth and value stocks explicitly over time. Our model exploits the affine structure of the Wishart process and admits analytically tractable solutions.

The rest of the paper is organized as follows. Section I describes the data source. In Section II, we define the SRR and LRR components within our model framework. We then use the empirical estimates of SRR to run predictive regressions, and we study the SRR and LRR jointly via GMM. In Section III, we introduce an extended LRR model that is flexible enough to account for the salient features of the data. We then proceed to calibrate the model to real data in Section IV. In Section V, we perform some robustness checks. Section VI concludes. All proofs are delegated to Appendix B.

# I. Data Source

In this section, we describe the data sources and summarize properties of returns, dividends and price-dividend ratios. All nominal quantities are deflated by CPI-U, which is published by the U.S. Bureau of Labor Statistics (BLS). We approximate the market portfolio by the value-weighted index from Center for Research in Security Prices (CRSP). Book-to-market portfolios are constructed in the same way as in Fama and French (1992). We use portfolio returns with and without dividends from the Kenneth R. French data library[5] to construct dividends. We also construct cash payouts adjusting for repurchases in the same way as Bansal et al. (2005a).[6]

The three book-to-market portfolios are the value-weighted stocks with the book-to-market ratio in the lower $0\% - 30\%$ percentiles (growth stocks), middle 30% and 70% percentiles and upper $70\% - 100\%$ percentiles (value stocks). We construct book-to-market portfolios every end of June by sorting stocks with their ratio of book values from the end of the last fiscal year and market values from last calendar year.

We set the end of year price of a portfolio by its price in December of the year. Data on returns and dividends are aggregated annually to keep them in line with annually updated macro variables,

[5]See `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

[6]We focus on portfolio dividends as the measure of portfolio cash-flow payout in the majority of this paper. However, in the robustness check section, we study alternative cash-flow measures where share repurchases are considered. To account for repurchases, we adjust monthly equity returns net of stock payouts by $\frac{P_{t+1}}{P_t} \min[\frac{n_{t+1}}{n_t}, 1]$, where $n_t$ is the number of shares after adjusting for splits, stock dividends using the CRSP share adjustment factor.

similar to Bansal and Yaron (2004) and Campbell and Cochrane (1999). The annual real log return is the sum of monthly real log returns. We calculate monthly dividends before seasonal adjustments from the difference of returns with and without dividends in the same way as Beeler and Campbell (2012). To adjust for the seasonality of dividends, the adjusted monthly dividend is the moving average of dividends in the previous 12 months. Dividend growth is calculated as the seasonally adjusted dividend in the current month divided by that in the previous month. To calculate the end-of-year price-dividend ratio, we divide the asset price by the sum of last 12 months of unadjusted dividends. The nominal 3-month Treasury bill rate data are taken from CRSP Fama risk-free rates. Given that future inflation is uncertain, we approximate the risk-free rate by the ex-ante real 3-month Treasury rate. Similar to Beeler and Campbell (2012), the ex-ante real 3-month Treasury rate is the fitted value by the regression of the ex-post real rate (deflated 3-month Treasury rate using realized inflation) on the nominal 3-month interest rate and the growth of CPI in the previous year.

## II. Short-run and long-run risk

In this section, we formally define SRR and LRR and study their empirical properties.

### A. *Identifying short-run risk*

To clarify the definition of SRR in our model, we start with a simplified long-run risk model for consumption and dividend growth dynamics. By denoting aggregated consumption by $C_t$ and by $D_t^i$ the dividend of asset $i$, we specify

$$\frac{dC_t}{C_t} = (\mu_c + X_t)dt + \sigma_{c,t} dB_t^c, \tag{1}$$

$$\frac{dD_t^i}{D_t^i} = (\mu_i + \phi_i X_t)dt + \sigma_{i,t} dB_t^i, \tag{2}$$

where $\mu_c$ and $\mu_i$, are constants, and $\sigma_{c,t}$, and $\sigma_{i,t}$ are possibly time-varying volatilities. The Brownian motions $B_t^c$ and $B_t^i$ may be correlated. $X_t$ is the LRR component in consumption and dividend growth, which is not correlated with $B_t^c$ and $B_t^i$.[7] The leverage parameter is $\phi_i$, which controls the

[7] We deliberately omit the specification for the dynamics of $X_t$ at this stage because the definition and measurement of SRR do not depend on the dynamics of $X_t$.

long-term comovement between consumption and dividend growth.

Given this specification, we define the short-run risk component in consumption, $\mathrm{SRR}^c$, as the time-$t$ realized integrated variance of consumption growth over one time period; that is,

$$\mathrm{SRR}^c_{t-1,t} := \int_{t-1}^{t} \frac{dC_s}{C_s}\frac{dC_s}{C_s}ds. \tag{3}$$

Similarly, the short-run risk component in asset $i$, $\mathrm{SRR}^i$, at time $t$ is defined as the realized integrated covariance:

$$\mathrm{SRR}^i_{t-1,t} := \int_{t-1}^{t} \frac{dC_s}{C_s}\frac{dD^i_s}{D^i_s}ds. \tag{4}$$

In our empirical analysis, we assume that one time period corresponds to one year. The above definitions of SRR reflect the (co)variation of the transient components in the consumption and dividend dynamics, as the following equations suggest:

$$\int_{t-1}^{t} \frac{dC_s}{C_s}\frac{dC_s}{C_s}ds = \int_{t-1}^{t} \sigma^2_{c,s}dt; \tag{5}$$

$$\int_{t-1}^{t} \frac{dC_s}{C_s}\frac{dD^i_s}{D^i_s}ds = \int_{t-1}^{t} \sigma_{c,s}\sigma_{i,s}dB^c_s dB^i_s. \tag{6}$$

There are two advantages of this definition. First, we can estimate the SRRs from realized variances and covariances directly from data. Second, because the LRR component $X_t$ only enters in the drift terms of the consumption and dividend dynamics, it does not interfere with the estimation of the SRRs.

Note that the reduced long-run risk model in this section could not capture the time variation of the SRR in asset $i$ because we impose no structures on the covariation with transient consumption shocks $\sigma_{c,s}\sigma_{i,s}dB^c_s dB^i_s$. Apart from stochastic variances, the covariation of the Brownian motions in consumption and cash flows are also time-varying. This calls for a more advanced model of the stochastic covariance, which is proposed in Section III.

Given the availability of monthly consumption and dividend data, we have ample data to estimate the SRR empirically. However, we face one obstacle—monthly and quarterly aggregated consumption data are seasonally adjusted, whereas dividends are adjusted by taking the yearly moving average. Hence, contemporaneous shocks in dividends and consumption are not reflected in these adjusted time series, which leads to bias in the SRRs estimates. To circumvent this problem, we use as monthly consumption growth the growth rate of the 12-month moving-average of the monthly aggregated consumption, which has some advantages. First, it fits well our model specification in which $C_t$ represents the aggregated consumption in the past year. In particular, under this construction the sum of the monthly consumption growth rates is equal to the growth rate of the annually aggregated consumption. Second, most of the literature uses seasonally adjusted dividends by taking the 12-month moving average. Seasonalizing consumption in a similar way enables us to calculate the covariance between consumption and dividend growth rates more accurately. We refer the reader to Appendix C for more details about the construction of monthly consumption growth.[8]

Following Equations (3) and (4), we directly estimate SRRs from data:

$$\mathrm{SRR}^c_{t-1,t} = \sum_{k=0}^{k=11} \left( \Delta c_{t+kh,t+(k+1)h} - \frac{1}{12} \sum_{j=0}^{j=11} \Delta c_{t+jh,t+(j+1)h} \right)^2, \tag{7}$$

$$\begin{aligned} \mathrm{SRR}^i_{t-1,t} = \sum_{k=0}^{k=11} & \left( \Delta c_{t+kh,t+(k+1)h} - \frac{1}{12} \sum_{j=0}^{j=11} \Delta c_{t+jh,t+(j+1)h} \right) \\ & \times \left( \Delta d^i_{t+kh,t+(k+1)h} - \frac{1}{12} \sum_{j=0}^{j=11} \Delta d^i_{t+jh,t+(j+1)h} \right), \end{aligned} \tag{8}$$

where $h = 1/12$, $i = v, g$ denote value or growth stocks, and $\Delta c_{t,t+h} = \log(C_{t+h}/C_t)$.

[Figure 1 about here.]

In Figure 1, we plot the evolution of SRRs over time. The SRRs seem to vary with the business cycle. Panel A shows the time-varying SRRs in growth and value stocks, and Panel B the SRR in consumption.

We find the SRRs vary with the business cycle. Figure 1 plots the evolution of SRRs over time. Panel A plots the time-varying SRRs in growth and value stocks, and Panel B the SRRs in consumption. In Panel A, the spikes (troughs) of the SRRs in value (growth) stocks seem to roughly coincide with NBER recorded recessions. This observation is consistent with Koijen et al. (2017). When macroeconomic activity is low, value stocks comove strongly with the business cycle and pay little dividends but growth stocks perform relatively well. If consumption growth is a

[8] BEA only publishes seasonalized monthly consumption using X13-ARIMA-SEATS. In Appendix C, we also show this algorithm does not affect the 12-month moving average materially.

measure of economic activity, then the cash flows of value (growth) stocks should move in the same (opposite) direction of the consumption growth rates around recessions. In Panel B, the SRR in consumption shoots up around economic recessions. This observation is in line with the common conception that the consumption volatility is countercyclical.

### *B. Regression results*

The results in Section II.A suggest that SRRs vary with the business cycle. To study the link between the SRRs and the business cycle formally, we use SRRs to predict future returns at horizons most pertinent to the business cycle. If SRRs convey bad (good) news about the economy, then they should predict negative (positive) future returns at business cycle horizons. Because the value premium is counter-cyclical, we expect the SRRs to have opposite implications for the future value-minus-growth returns.

The predictive regressions are of the following form:

$$r_{t,t+h} = \beta_0 + \beta_c \mathrm{SRR}^c_{t-1,t} + \beta_{cg}\mathrm{SRR}^g_{t-1,t} + \beta_{cv}\mathrm{SRR}^v_{t-1,t} + \boldsymbol{\beta}_z' Z_t + \epsilon_{t,t+h}, \tag{9}$$

where $\mathrm{SRR}^c$, $\mathrm{SRR}^g$ and $\mathrm{SRR}^v_{t-1,t}$ are SRRs in consumption, growth stocks and value stocks, $Z_t$ is a vector of additional optional predictors, which are included to determine the robustness of SRRs in predictive regressions, and $\epsilon_{t,t+h}$ is the residual. The LHS of Equation (9) is either future market excess returns or future value-minus-growth returns.

To test whether the predictive power of SRRs in the regression (9) are already contained in the macroeconomic variables, we include additional macroeconomic variables as predictors. These variables include the approximate log consumption-wealth ratio (*cay*, Lettau and Ludvigson, 2001a), income-consumption ratio ($I/C$, Santos and Veronesi, 2006) and Cochrane-Piazessi factor (CP, Cochrane and Piazzesi, 2005). We also study the log price-dividend ratio $\log \frac{P}{D}$.

The correlations of the independent variables are shown in Table I. The variables CP and $\log \frac{P}{D}$ are strongly correlated, and both are negatively correlated with the income-consumption ratio. These variables capture similar aspects of the business cycle: a rosy economic outlook associates with a large CP factor, a large price-dividend ratio and a small income-consumption ratio; while an adverse economic outlook is vice versa. The income-consumption ratio is negatively correlated with *cay* because both are related to the consumption-wealth ratio of the representative agent. The *cay* is pro-cyclical, which is in line with the argument in Lettau and Ludvigson (2001a) that the representative agent consumes a larger share out of her total wealth in anticipation of good portfolio returns. $\mathrm{SRR}^c$ is correlated with the income-consumption ratio and is negatively correlated with *cay* and the price-dividend ratio, which implies that $\mathrm{SRR}^c$ is negatively correlated with the consumption-wealth ratio. The SRR in value stocks is negatively correlated with that in growth stocks, which is consistent with the observations that these two variables move in opposite directions during recessions.

[Table I about here.]

### B.1. In-sample predictability

Figure 2 shows the adjusted $R^2$ of the predictive regressions at different horizons using Equation (9). For future market excess returns, forecasting regressions using past variables has the best predictive power at the four-quarter horizon, where the $R^2$ is 17.6%. Beyond the business cycle horizons, predictability wears off. This supports our hypothesis that SRRs capture business cycle risks, which do not persist over the long term. For the value-minus-growth returns, predictability increases over time, where $R^2$ continues to rise to 11.5% at 12-quarter horizon.

[Figure 2 about here.]

[Table II about here.]

[Table III about here.]

To test the significance of SRRs in the presence of other macroeconomic variables as predictors, we focus on the four-quarter horizon for the market excess returns and 12-quarter horizon for the value-minus-growth returns. The horizons are chosen to maximize predictability. Table III and Table II report the regression results using SRRs and macroeconomic variables as predictors.[9]

[9] We have further tested those predictive regressions using IVX estimation (Kostakis et al., 2014). The predictive regression on market excess returns is significant at the 99% level, and the regression on value-minus-growth returns is significant at the 90% level.

A large consumption and value stock SRR, or a small growth stock SRR, usually accompanies an adverse economic outlook in the future one-year and the market excess return drops. The adjusted $R^2$s using SRRs are even larger than those of in-sample constructed *cay* in both samples. If we combine the predictors, then $R^2$ would increase more and *cay* would remain significant. This suggests that SRRs are not redundant, even when given predictors such as *cay*. Other predictors—including $\log(\frac{P}{D})$, CP factor and income-consumption ratio—manifest little forecasting power of the future market excess returns in the one-year horizon. Consistent with the negative sign of the income-consumption ratio, which is first documented in Santos and Veronesi (2006), SRR$^m$ is positively associated with the future market excess returns. To supplement this study, we also ran predictive regressions using univariate SRRs. Univariate regressions of the future market excess returns on SRRs gave coefficients similar to those in the multivariate regressions. This implies that SRRs carry orthogonal information for the equity risk premia. The invariance of regression coefficients in univariate and multivariate regressions provides additional confidence in the robustness of our regression results.

The finding that SRRs in consumption negatively correlate with future market excess returns seem counter-intuitive at first glance because common wisdom would suggest that the market risk premium is positively correlated with market return volatility. However, consumption volatility is not perfectly correlated with all of the components in the covariance matrix of the cross-section of assets. Indeed, the correlation between the SRR in growth stocks and the SRR in consumption is almost zero, while that between the SRR in value stocks and the SRR in consumption is nontrivial. The regression results suggest that the information in the cross section of SRRs plays a vital role beyond the univariate market volatility.

For the future value-minus-growth excess returns, the SRR in consumption is significant. The SRRs in value and growth stocks do not explain the variations in the value-minus-growth excess returns. The negative coefficient sign suggests the value premium is counter-cyclical, which is for example consistent with Zhang (2005). Indeed, the quarterly aggregated market excess returns and value-minus-growth returns have a negative correlation of $-16.15\%$. The value-minus-growth returns tend to rise post-crisis, and are typically associated with a spike in the SRRs in consumption. This observation is in line with Avramov et al. (2013), such that the value premium is mainly derived from survived distressed firms that are valued lower and bounce back harder post-crisis. Because recoveries last longer than recessions, the adjusted $R^2$ in forecasting the future value-minus-growth excess returns increases over time.

### B.2. OOS predictability and market timing strategy

Welch and Goyal (2008) argue that most predictive regressions cannot beat the historical average in forecasting the OOS market excess returns. In contrast, we find that our predictive regressions using SRRs have out-of-sample explanatory power. This out-of-sample predictability gives rise to an out-of-sample market-timing strategy, which leads to an economically significant improvement in portfolio performance for mean-variance investors. Moreover, the predictive power of SRRs has been high in the last 20 years, the period in which Welch and Goyal (2008) find that most documented predictors have poor predictive power in both in-sample and out-of-sample.

We consider the out-of-sample $R^2$, Sharpe ratio and the cumulated excess returns corresponding to four kinds of predictive regressions—three univariate regressions using SRRs in consumption, growth and value stocks, respectively; and a multivariate regression using all SRRs given above—to forecast the future one-year market excess returns. At the start of each year, we estimate the regression coefficients using data from the previous 35 years.[10] We then compare our forecasts and the historical average to calculate out-of-sample $R^2$:

$$R_{OS}^2 = 1 - \frac{\sum_{t=1}^{T}(r_{t,t+1}^e - \hat{r}_{t,t+1}^e)^2}{\sum_{t=1}^{T}(r_{t,t+1}^e - \bar{r}_t^e)^2} \tag{10}$$

where $\bar{r}_t^e$, measured at beginning of year $t$, is the historical average of the annual market excess returns in the past 35 years.

For each predictive regression, we construct a market-timing strategy. At the beginning of each year, the regression gives an out-of-sample estimate of the market excess of the following year, and we set the estimate as the weight to adjust the position in the market equity premium.[11] To make returns in such zero net position strategies comparable to the market excess returns, we ex post

[10]We estimate the regression coefficients from a quarterly sample. To avoid using future information, observations after the beginning of last year are excluded.

[11]Given CRRA utility and constant variance in market excess returns, and supposing that the investor choose from a risk-free asset and a market portfolio to invest, her position in the market portfolio is proportional to its estimated return.

scale weights of these strategies such that their returns have ex post the same volatility as the market excess returns. Table IV reports the out-of-sample $R^2$s of predictive regressions and the Sharpe ratios of corresponding market-timing strategies. Note that scaling positions of a strategy ex post have no effect on the Sharpe ratio.

[Table IV about here.]

The out-of-sample $R^2$s of predictive regressions using individual and all SRRs are positive, which suggests that SRRs can robustly forecast the future one-year market excess returns. The Sharpe ratios of the market-timing strategies give more evidence to support the out-of-sample explanatory power in the market excess returns. Compared with the Sharpe ratio of the market excess returns, the Sharpe ratio of any market-timing strategy based on a single SRR measure is improved by about 60% and the Sharpe ratio of the strategy using all SRRs is doubled.

[Figure 3 about here.]

Figure 3 plots the cumulative returns using different investment strategies starting at the end of 1995. Market-timing strategies adjust positions in the market and the risk-free asset only once every year. Unlike many portfolio selection strategies, where the bid-ask spread could eat up a significant part of the profits, this market-timing strategy is almost free from such costs. Moreover, the coefficients on rolling predictive regressions always have the same sign as their in-sample counterparts.

In summary, the predictive power remains strong, even in out-of-sample regressions, which leads to a reasonable improvement in the asset allocation strategies for a mean-variance investor.

## *C. Estimating the LRR*

This section studies the properties of the LRR and the SRR jointly via GMM. Unlike the SRRs, the LRR component does not admit a time series of empirical estimates. However, with the help of SRR estimates, we can study the properties of the LRR component in consumption growth dynamics via GMM. The results in this section only depend on consumption and dividend growth rates, and are thus independent of any assumption on asset prices.

Although $X_t$ is highly persistent and changes little over shorter periods, $X_t$ could potentially vary a lot from year to year. From Equation (1), we obtain the unconditional variance of the annually aggregated consumption growth as,

$$\begin{aligned}
\mathrm{Var}\left(\int_{t-1}^{t} \frac{dC_s}{C_s} ds\right) &= \mathrm{Var}\left(\int_{t-1}^{t} X_s ds\right) + \mathbb{E}\left((\int_{t-1}^{t} \sigma_{c,s} dB_s^c)^2\right) + \mathbb{E}\left((\int_{t-1}^{t} \sigma_{c,s} dB_s^c)(\int_{t-1}^{t} X_s ds)\right) \\
&= \mathrm{Var}\left(\int_{t-1}^{t} X_s ds\right) + \mathbb{E}\left(\int_{t-1}^{t} \sigma_{c,s}^2 ds\right) \qquad (11) \\
&= \mathrm{Var}(\int_{t-1}^{t} X_s ds) + \mathbb{E}(\mathrm{SRR}_{t-1,t}^c), \qquad (12)
\end{aligned}$$

where the cross term between $X_t$ and $B_s^c$ is zero because they are uncorrelated. Thus, the variance of the annually aggregated consumption growth is the sum of the variance of integrated LRR and the expectation of SRR. Similarly, the contribution of LRR to the covariance between dividend and consumption growth is controlled by $\phi_i$, as illustrated in Equation (13)

$$\mathrm{Cov}\left(\int_{t-1}^{t} \frac{dC_s}{C_s} ds, \int_{t-1}^{t} \frac{dD_s^i}{D_s^i} ds\right) = \phi_i \,\mathrm{Var}(\int_{t-1}^{t} X_s ds) + \mathbb{E}(\mathrm{SRR}_{t-1,t}^i) \qquad (13)$$

Motivated by Equations (12) and (13), we can study the variance of integrated LRR by GMM. At the end of each year, we can estimate the annually aggregated growth of consumption and dividends $\Delta c_{t-1,t} := \int_{t-1}^{t} \frac{dC_s}{C_s} ds$, $\Delta d_{t-1,t}^i := \int_{t-1}^{t} \frac{dD_s^i}{D_s^i} ds$, and SRRs as the realized variances or covariances. For notational brevity, we denote the variance of the annually integrated LRR and the mean of SRRs in consumption by

$$\sigma_X^2 := \mathrm{Var}(\int_{t-1}^{t} X_s ds), \quad \mu_{\mathrm{SRR}^i} := \mathbb{E}(\mathrm{SRR}_{t-1,t}^i) \qquad (14)$$

Hence, we can formulate the GMM according to the following moment conditions:

$$\mathbb{E}(\Delta c_{t-1,t}) = \mu_c, \qquad \mathbb{E}(\Delta d_{t-1,t}^i) = \mu_i, \qquad (15)$$

$$\mathbb{E}(\Delta c_{t-1,t}^2) = \mu_c^2 + \sigma_X^2 + \mu_{\mathrm{SRR}^c}, \qquad \mathbb{E}(\Delta d_{t-1,t}^i \Delta c_{t-1,t}^2) = \mu_c \mu_i + \phi_i \sigma_X^2 + \mu_{\mathrm{SRR}^i}, \qquad (16)$$

$$\mathbb{E}(\mathrm{SRR}_{t-1,t}^c) = \mu_{\mathrm{SRR}^c}, \qquad \mathbb{E}(\mathrm{SRR}_{t-1,t}^i) = \mu_{\mathrm{SRR}^i}. \qquad (17)$$

where $i = v, g, m$ represents the value, growth or market portfolio.

[Table V about here.]

The GMM estimation results are reported in Table V. We test all of the parameters with the null hypothesis that it equals zero against the alternative that it is larger than zero. The $\sigma_X^2$ is significantly different from zero at the 99.9%-level, which suggests the existence of a nontrivial LRR component. Consistent with the previous literature, such as Bansal et al. (2005a), the value stocks load more LRR than the growth stocks. The leverage parameter of value stocks $\phi_v$ is significantly larger than zero at 99%-level, but the leverage parameter of growth stocks $\phi_g$ is not significantly larger than 0. Apart from LRR, our paper also identifies that value stocks have higher SRR than the growth stocks, with the mean of SRR in value stocks significantly larger than zero at the 99%-level.

[Table VI about here.]

To further study the properties of the LRR, we summarize the statistics of the SRRs and consumption growth in Panel A of Table VI. The means of SRRs are smaller than the covariances between the annually aggregated consumption and cash flows growth rates, which confirms the existence of LRR in consumption and dividends of book-to-market portfolios. The first order autocorrelation of the monthly consumption growth rates over the whole sample is 92.4%. The high persistence could be explained by the persistent LRR component. The autocorrelation of the annually aggregated consumption growth rates is 47.9%, which is smaller than the autocorrelation for the monthly growth rates because the LRR $X_t$ varies more over a longer period. Given the almost constant LRR component within a year, within year autocorrelation of monthly aggregated growth rates almost removes the LRR part as the mean and only considers the transient component of the consumption growth. Therefore, within-year first-order autocorrelation of monthly aggregated growth rates should be smaller. Indeed, the first-order autocorrelations of the monthly aggregated consumption growth rates calculated within each year are small and volatile, with the mean 35.5% and the standard deviation 26.5%.

To verify that our approach does not falsely detect a persistent and large LRR component, we simulate consumption growth processes under different assumptions regarding the LRR component.

Our results are summarized in Panel B of Table VI. Under a persistent and large LRR component, the simulated consumption growth process has similar patterns in the statistics: the variance of the annually aggregated growth rates is larger than the sample mean of the SRRs, and the GMM estimation rejects the LRR component at zero. However, if the persistence of LRR component is zero or the variance of LRR component is zero such that there are only transient shocks, then the SRRs are almost as large as the variance of annually aggregated consumption growth, and the GMM estimation cannot reject the variance of persistent LRR at zero. Further details about the simulation exercise can be found in Appendix D.

## III. The model

In this section, we introduce the model, and we derive solutions to generate the patterns in asset prices and SRRs.

### *A. Model setup*

We formulate our economy in continuous time and we equip our representative agent with the recursive utility function, as defined in Duffie and Epstein (1992). We depart from the previous literature on LRR models in how we incorporate fluctuating economic uncertainty.

To model the covariance structure of the transient shocks in consumption and dividend growth, we impose a matrix-valued Wishart process given by

$$d\Sigma_t = (kQQ' + M\Sigma_t + \Sigma_t M')dt + \sqrt{\Sigma_t} dB_t^\sigma Q + Q' d(B_t^\sigma)' \sqrt{\Sigma_t}, \tag{18}$$

where $B_t^\sigma \in \mathbb{R}^{n\times n}$ is a matrix of independent Brownian motions. The constant matrices $M \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ control the mean reversion and volatility of the Wishart process.[12] To maintain parsimony, we fix the long-term mean for $\Sigma_t$ to $kQQ'$ and we set the scalar $k = n + 1$.[13]

As in Bansal and Yaron (2004), we let both dividend and consumption growth be characterized by a persistent LRR component $X_t$, which follows a mean-reverting process with stochastic

[12] To guarantee stationarity, we assume $M$ to be negative definite. For $Q \in \mathbb{R}^{n\times n}$, we impose symmetry and positive definiteness to reduce the number of parameters in our estimation. These restrictions on $Q$ are without loss of generality.

[13] This is a sufficient condition for $\Sigma_t$ to stay positive definite, see Mayerhofer et al. (2011).

volatility,

$$dX_t = -\alpha X_t dt + \underbrace{\sqrt{\delta_x' \Sigma_t \delta_x}}_{\in \mathbb{R}} dB_t^X, \tag{19}$$

where $\alpha$ controls the speed of mean-reversion, $\delta_x \in \mathbb{R}^n$ is a constant vector, and $B_t^X \in \mathbb{R}$ is a Brownian motion. Note that the volatility of the transient shock $\sqrt{\delta_x' \Sigma_t \delta_x}$ is univariate, despite being a function of the stochastic matrix $\Sigma_t \in \mathbb{R}^{n \times n}$.

Our economy models $n$ portfolios jointly. Each portfolio pays out dividends $D_t^i$, $i = 1, \ldots, n$ with the following dynamics,

$$\frac{dD_t^i}{D_t^i} = (\mu_i + \phi_i X_t)dt + \underbrace{\delta_i' \sqrt{\Sigma_t} dB_t}_{\in \mathbb{R}} + \sigma_i dB_t^i, \tag{20}$$

where $B_t \in \mathbb{R}^n$ is a vector of Brownian motions shared by all firms, $B_t^i$ is univariate Brownian motion for firm $i$, $\sigma_i$ is the volatility of firm-specific shock, $\mu_i$ measures the mean of firm $i$'s dividend growth process, $\delta_i \in \mathbb{R}^n$ is a constant vector, and $\phi_i$ measures its loading on the LRR component $X_t$.

To generate a time-varying correlation between consumption growth and dividend growth, we link the stochastic covariance matrix to the consumption process $C_t$. We assume that $C_t$ has the following dynamics:

$$\frac{dC_t}{C_t} = (\mu_c + X_t)dt + \underbrace{\delta_c' \sqrt{\Sigma_t} dB_t}_{\in \mathbb{R}} + \sigma_{c,t} dB_t^c, \tag{21}$$

where $B_t^c \in \mathbb{R}$ is a Brownian motion independent of $B_t$, $\mu_c$ is the mean consumption growth rate, and the constant vector $\delta_c \in \mathbb{R}^n$ together with $\Sigma_t$ controls the loading on the transient component of consumption growth. Our representative agent may not generate income solely from dividends, but may also generate income from other sources, such as labor. Hence, we add an additional source of risk in the consumption growth dynamics, $\sigma_{c,t} dB_t^c$, which is not spanned by asset markets. we specify

$$\sigma_{c,t}^2 = \bar{\sigma}_c^2 + \mathrm{Tr}(\chi_c \Sigma_t), \tag{22}$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a square matrix. We assume that the Brownian motions $B_t$, $B_t^X$, $B_t^i$ and $B_t^c$ are mutually independent.

Our specifications in Equations (20) and (21) are consistent with the LRR framework in Equations (1) and (2), where a Wishart process models the multivariate stochastic volatility structure for consumption and dividend growth rates. In particular, we have

$$\mathrm{Var}_t\left(\frac{dC_t}{C_t}\right) = \mathrm{Tr}(\delta_c\delta_c'\Sigma_t)dt + \sigma_{c,t}^2 dt = \mathrm{Tr}((\delta_c\delta_c' + \chi_c)\Sigma_t)dt + \bar{\sigma}_c^2 dt, \tag{23}$$

$$\mathrm{Cov}_t\left(\frac{dC_t}{C_t}, \frac{dD_t^i}{D_t^i}\right) = \mathrm{Tr}(\delta_i\delta_c'\Sigma_t)dt, \quad i = 1, \ldots n, \tag{24}$$

The vectors $\delta_c$ and $\delta_i$ determine how much the variances and covariances load on the different elements of the matrix $\Sigma_t$. Furthermore, we can construct the theoretical counterparts of Equations (3) and (4) to accommodate the time-varying SRR in closed-form:

$$\mathrm{SRR}^i = \int_{t-1}^{t} \underbrace{\delta_c'\Sigma_s\delta_i}_{\in\mathbb{R}} ds, \tag{25}$$

$$\mathrm{SRR}^c = \int_{t-1}^{t} \left(\mathrm{Tr}((\delta_c\delta_c' + \chi_c)\Sigma_t) + \bar{\sigma}_c^2\right) ds. \tag{26}$$

The SRR in asset dividends only captures the common shocks between consumption and dividends through $\delta_c$ and $\delta_i$. Nonetheless, the SRR in consumption includes an additional component from $\sigma_{c,t}$ as defined in Equation 22. In models with univariate dividend and consumption growth variances, a larger consumption growth variance accompanies larger expected returns. In our model, dividend growth has a stochastic covariance structure, and the loadings $\chi_c$ allow the consumption volatility to load flexibly on the components in the cash flow covariance matrix $\Sigma_t$, which is crucial in replicating the negative relationship between SRR in consumption and future asset returns.

## B. *Model solutions*

We follow Duffie and Epstein (1992) and assume that the representative agent has recursive preferences. The value function $J$ satisfies

$$J_t = \max_{C_s} \mathbb{E}_t\left[\int_t^T f(C_s, J_s)ds\right], \tag{27}$$

where

$$f(C_t, J_t) = \begin{cases} \beta\theta J_t[(\frac{C_t}{((1-\gamma)J_t)^{1/(1-\gamma)}})^{1-\frac{1}{\psi}} - 1] & \text{if } \psi \neq 1,\ \theta = \frac{1-\gamma}{1-1/\psi}, \\ \beta(1-\gamma)J_t \log\left(\frac{C_t}{((1-\gamma)J_t)^{1/(1-\gamma)}}\right) & \text{if } \psi = 1. \end{cases} \tag{28}$$

where $\gamma$ denotes the risk aversion coefficient and $\psi$ denotes the intertemporal elasticity of substitution. We assume that the representative agent prefers early resolution of risk, such that $\gamma > 1$

and $\psi > 1$.[14] To solve the model, we make use of the log-linear approximation as in Campbell and Shiller (1988). Thus, we obtain a quasi-closed-form solution up to generalized continuous-time algebraic Riccati equations (CARE).[15]

PROPOSITION 1: *The value function is given by*

$$J(W_t, X_t, \Sigma_t) = \exp(A_0 + A_1 X_t + \mathrm{Tr}(A_2 \Sigma_t)) \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{29}$$

*and the consumption-wealth ratio is given by*

$$\frac{C_t}{W_t} = \beta^{\psi} \exp(A_{0a} + A_{1a} X_t + \mathrm{Tr}(A_{2a} \Sigma_t)), \tag{30}$$

*where* $A_{ka} = \frac{1-\psi}{1-\gamma} A_k$, *for* $k = 0, 1, 2$. *Furthermore,*

$$\begin{aligned} A_0 &= \frac{1}{g_1 \psi}(\theta(g_1 - g_1 \log g_1 + g_1 \psi \log \beta) - \beta\theta + \psi \, \mathrm{Tr}(A_2 \Omega \Omega') + (1-\gamma)\mu_c - \frac{(1-\gamma)\gamma}{2} \bar{\sigma}_c^2) \quad (31) \\ A_1 &= \frac{1-\gamma}{(g_1 + \alpha)\psi}, \end{aligned}$$

*where* $g_1 = \exp(\mathbb{E}(c_t - w_t))$, $c_t := \log C_t$ , $\theta = \frac{1-\gamma}{1-1/\psi}$ *and* $w_t := \log W_t$. *The term* $g_1$ *and the constant positive semidefinite symmetric matrix* $A_2$ *need to be solved by generalized CARE.*

Some comments are in order here. First, the generalized CARE admits a positive semidefinite solution with reasonable computational efficiency. Hence, although some numerical calculations are required, the model is still highly tractable. Second, with $\gamma > 1$ and $\psi > 1$, we have $A_{1a} < 0$. Therefore, following the interpretations of the standard LRR model, the representative agent reacts to positive news in long-term consumption growth $X_t$ by consuming less out of her wealth portfolio, thereby smoothing consumption. Consequently, the substitution effect dominates the income effect. Third, $A_2$ is positive semidefinite. Therefore, the consumption-wealth ratio increases when an overall increase of variance occurs, similar to Bansal and Yaron (2004). However, because in our model each element of the stochastic covariance matrix could affect the consumption-wealth ratio through $A_2$, elements of the covariance matrix have heterogeneous effects on consumption-wealth ratio. Finally, the effect of the persistent component $X_t$ on the consumption-wealth ratio $A_1$ increases with the persistence of $X_t$, which in turn is inversely related to the mean reversion

[14] We discuss the case $\psi = 1$ in Appendix B.

[15] The potential errors introduced by the log-linear approximation have come under scrutiny in a recent paper by Pohl et al. (2018). We perform some robustness checks in Section V and we find that the errors induced by log-linear approximations are negligible.

coefficient $\alpha$.

PROPOSITION 2: *The state price deflator follows the dynamics*

$$\frac{d\pi_t}{\pi_t} = -r_f dt - \Lambda dB_t - \Lambda^c dB_t^c - \Lambda^X dB_t^X - \mathrm{Tr}(\Lambda^\sigma dB_t^\sigma), \tag{32}$$

*with*

$$\Lambda = \gamma \delta_c' \sqrt{\Sigma_t}, \quad \Lambda^c = \gamma \sigma_{c,t}, \quad \Lambda^X = -\frac{1-\psi\gamma}{1-\gamma} A_1 \sqrt{\delta_x' \Sigma_t \delta_x}, \quad \Lambda^\sigma = -2\frac{1-\psi\gamma}{1-\gamma} Q A_2 \sqrt{\Sigma_t}. \tag{33}$$

*Furthermore, the risk-free interest rate is given as*

$$r_f = r_0 + r_x X_t + \mathrm{Tr}(r_\Sigma \Sigma_t), \tag{34}$$

*where the expressions for the coefficients $r_0$, $r_x$, and $r_\Sigma$ are given in equations* (B26) *to* (B28).

From Equation (33), we can identify four components for the market price of risk in our model. The first two components, $\Lambda$ and $\Lambda^c$, are the market prices of risk on transient consumption shocks, where $\Lambda^c$ arises from the additional source of risk that is not spanned by the asset market. These two components are proportional to the risk aversion coefficient $\gamma$ and they do not depend on the intertemporal elasticity of substitution $\psi$. The third component, $\Lambda^X$, is the market price of risk for exposure to innovations in LRR. The fourth component, $\Lambda^\sigma$, represents the market price of risk for innovations in the Wishart covariance process.

Our specification of the market price of risk extends the previous LRR models in that we account not only for the variance risk as in Zhou and Zhu (2015) but we also account for the covariance risk. The off-diagonal elements of the covariance matrix are needed to match the time-varying returns in the cross-section of assets.

Our LRR model generates a risk-free interest rate in Equation (34) as an affine function of the LRR component $X_t$ and elements of $\Sigma_t$. This specification of the risk-free rate is similar to the term structure models in Buraschi et al. (2008) and Cieslak and Povala (2016b). However, our focus is on the dynamics of the cross-section of equity returns instead of the risk-free rate term structure.[16]

[16]Note that if $\psi\gamma = 1$, then the utility function reduces to CRRA form. Under CRRA, uncertainty in the future utility arising from uncertainty in the consumption growth process is no longer priced, so $\Lambda^X$ and $\Lambda_\sigma$ are zero. Furthermore, we would obtain $r_\Sigma = 0$, which shuts down the major channel of variation in risk-free interest rate because $X_t$ moves only slowly.

PROPOSITION 3: *The dividend-price ratio for asset i has the following form*

$$\frac{D_t^i}{P_t} = \exp(A_{0i} + A_{1i}X_t + \mathrm{Tr}(A_{2i}\Sigma_t)), \tag{35}$$

*$A_{0i}$ and $A_{1i}$ are given by*

$$A_{0i} = \frac{1}{g_{1i}}(-\mu_i + \mathrm{Tr}(A_{2i}\Omega\Omega') - g_{0i} + r_0), \tag{36}$$

$$A_{1i} = -\frac{\phi_i - \frac{1}{\psi}}{g_{1i} + \alpha}, \tag{37}$$

*where $g_{1i} = \exp(\mathbb{E}(d_t^i - p_t^i))$, $d_t^i = \log D_t^i$, $p_t^i = \log P_t^i$, and $A_{2i}$ is a symmetric positive semidefinite matrix of coefficients. The expressions for $g_{1i}$ and $A_{2i}$ need to be solved by generalized CARE. The equity risk premium for asset i is*

$$-\frac{d\pi_t}{\pi_t}\frac{dP_t^i}{P_t^i}/dt = \gamma\delta_c'\Sigma_t\delta_i + \psi\left(1 - \frac{1}{\theta}\right)A_1A_{1i}\delta_x'\Sigma_t\delta_x + 4\psi\left(1 - \frac{1}{\theta}\right)\mathrm{Tr}(QA_{2i}\Sigma_tA_2Q). \tag{38}$$

$A_{1i}$ and $A_{2i}$ play similar roles to $A_1$ and $A_2$. If $\phi_i > 1/\psi$, then we have $A_{1i} < 0$ and an increase in the LRR component $X_t$ drives up the valuation of asset $i$. In other words, the substitution effect dominates the income effect. Because $A_{2i}$ is positive semidefinite, an increase in overall volatility in $\Sigma_t$ drives down the valuation of asset $i$.

The equity risk premium in Equation (38) comprises three parts. The first part is determined by the covariance of dividend growth and consumption growth, scaled by the risk aversion coefficient. A higher covariance implies a higher risk premium. The second part is the contribution of the variation in the LRR component. Higher LRR volatility or intertemporal elasticity of substitution leads to a larger equity premium. As Bansal and Yaron (2004) show, a sufficiently high persistence in the dynamics of LRR component helps to generate a large equity premium. Under our assumption that the representative agent prefers early resolution of risk, $\gamma > 1$ and $\psi > 1$. Hence, a high persistence ( a low value for $\alpha$), leads to the large product $A_1A_{1i}$. Furthermore, because $\Sigma_t$ is positive definite, $\delta_x'\Sigma_t\delta_x$ is positive. Therefore, the risk premium part arising from long-run risk is always positive. The third part arises from the exposure to the innovations in transient consumption shocks, which captures the compensation for the SRR. In models without stochastic covariation, the correlation between shocks are constant and the only variation in this part stems from the stochastic volatility, which lacks flexibility to model the compensation from the SRR. In contrast, the Wishart process enables SRR to manifest its importance in the risk premium.

To avoid over-parametrization, we impose additional restrictions on the model. Assume that in the economy there are three portfolios: the market portfolio, the portfolio of growth stocks and the portfolio of value stocks. We further assume that the stochastic covariance matrix is a $2 \times 2$ Wishart process, which has three free components because any covariance matrix is symmetric. Under these restrictions, the risk premium for asset $i$ in Equation (38) is the linear combination of the three components in the Wishart process.

Recall from Equations (23) and (24) that the instantaneous variance of consumption growth and the instantaneous covariance between consumption growth and dividend growth are linear combinations of the components in the Wishart process $\Sigma_t$ and a constant. Hence, if $\left(\frac{dC_t}{C_t}\right)^2$, $\left(\frac{dC_t}{C_t}\frac{dD_t^g}{D_t^g}\right)$ and $\left(\frac{dC_t}{C_t}\frac{dD_t^v}{D_t^v}\right)$ are linearly independent, then any risk premium can be written as their combinations and a constant part. Therefore,

$$\begin{aligned} E_t(r_{e,t}^i) &= \gamma\delta_c'\Sigma_s\delta_i + \psi\left(1-\frac{1}{\theta}\right)A_1A_{1i}\delta_x'\Sigma_s\delta_x + 4\psi\left(1-\frac{1}{\theta}\right)\mathrm{Tr}(QA_{2i}\Sigma_sA_2Q) \\ &= \beta_0 + \beta_c\left(\frac{dC_t}{C_t}\right)^2 + \beta_{cg}\left(\frac{dC_t}{C_t}\frac{dD_t^g}{D_t^g}\right) + \beta_{cv}\left(\frac{dC_t}{C_t}\frac{dD_t^v}{D_t^v}\right) \end{aligned} \tag{39}$$

for some $\beta_0$, $\beta_c$, $\beta_{cg}$, $\beta_{cv}$ chosen to match the four dimensions in the $2 \times 2$ Wishart process and the constant. In other words, the model implies that the equity risk premia of an asset can be explained by the instantaneous variance of the consumption growth and the instantaneous covariance between the growth of consumption and the dividends in growth stocks and value stocks.

Equation (39) implies that the representative agent takes into account the time-varying covariance in the cash flows of assets, which leads to the time-varying equity risk premia. Given that SRRs are realized variances and covariances, the model could replicate the relation between SRRs and asset returns.

PROPOSITION 4: *The model-implied regression coefficients $\beta_0, \beta_c, \beta_{cg}, \beta cv$ can be derived in closed-form for predictive regressions of future returns on SRRs*

$$r_{t,t+\frac{Q}{4}} = \beta_0 + \beta_c SRR_{t-1,t}^c + \beta_{cg} SRR_{t-1,t}^g + \beta_{cv} SRR_{t-1,t}^v + \epsilon_{t,t+\frac{Q}{4}}, \tag{40}$$

*where $Q$ denotes quarters, $SRR_{t-1,t}^c$ and $SRR_{t-1,t}^g$ ($SRR_{t-1,t}^v$) are defined in Equations (3),(4), $\epsilon_{t,t+\frac{Q}{4}}$ is the residual. See Section B.4 for further details.*

# IV. Quantitative model results

In this section we aim to calibrate the parameters to match moments in the sample from 1959–2017,[17] and we will study the quantitative results implied by our model. More details about the calibrations and the derivations of model-implied moments are given in Appendix B.6.

To reflect the dynamics of cash flows, we match the unconditional mean, the first-order autocorrelation and the volatility of growth rates of consumption and dividends of value stocks, growth stocks and market portfolio. To ensure that our model captures patterns in asset returns, we also match the unconditional mean, the volatility and the first-order autocorrelation of the risk-free rate, the aggregated market equity premia, and equity premia of value and growth stocks, as well as their price-dividend ratios. In particular, to verify the additional pricing channel of SRR, we match the theoretical moments of SRRs with sample moments of SRRs. Moreover, we match regression coefficients in predictive regressions of future returns on SRRs. The model is calibrated by matching to these quantities jointly.

For a Wishart process of dimension $n$, $Q$ has $n(n+1)/2$ free parameters while $M$ has $n^2$ parameters. To reduce the number of parameters, we set $n = 2$. To avoid overidentification and further reduce the number of parameters, w.l.o.g. we specify[18]

$$\delta_m = (\delta_1^m, \delta_2^m)', \; \delta_g = (1, 0)', \; \delta_v = (0, 1)'. \tag{41}$$

where $\delta_m$, $\delta_g$, $\delta_v$ correspond to the market, growth and value portfolio, respectively. We restrict $M$ to be lower triangular, which further reduces the number of parameters. Table VII shows our baseline calibration.[19]

[Table VII about here.]

## *A. Consumption and cash-flow growth*

[Table VIII about here.]

[17] Monthly aggregated consumption data is not available before 1959.

[18] This restriction is without loss of generality. For a Wishart process $\Sigma_t$ with mean reversion $M$ and scale of shocks $Q$, $L\Sigma_t L'$ is still a Wishart process with $LML$ and $QL'$ replacing $M$ and $Q$. If $\delta_i$ are arbitrary, set $L' = (\delta_g, \delta_v)$ and we transform Wishart process so that our restriction in Equation 41 holds without changing any model implications.

[19] R package *DEoptim* (Mullen et al., 2011) is used to estimate the parameter values.

First, we study the dynamics of the growth rates of consumptions and dividends under the joint calibration. See Panel A of Table VIII for the mean, standard deviation and first-order autocorrelation of the growth rates of the annually aggregated consumption and dividends. Most model-implied first moments lie within one standard deviations from their sample counterparts. The volatility of the growth rates implied by the model are also close to the realized volatilities in sample. Our model also generates realistic first-order autocorrelations for dividend growth rates, where the model-implied values lie within one standard deviation from the sample counterparts. This model replicates the pattern that the first-order autocorrelation of consumption growth is larger than those of dividend growth, although the model-implied $AC1(\Delta c)$ seems larger than empirical value.

Our baseline calibration has $\alpha = 0.087$, which translates into a monthly persistence of the the consumption growth rate at 92.3%. The persistence in the monthly consumption growth is comparable to the persistence at 97.8% in BY model. The Wishart covariance matrix is mean reverting and is controlled by the mean reversion matrix $M$. $M$ has eigenvalues $-0.163, -0.0875$, so the Wishart covariance matrix has a monthly persistence between 98.7% and 99.3%. While the notion of half-life is not immediately applicable to the mean reversion matrix $M$, the half-lives implied from the eigenvalues of $M$ are between 2 and 7 years. The variation cycle of the Wishart covariance matrix has a similar length as a business cycle, which substantiates our claim that SRR is sensitive to business cycle risks.

We match both the correlations of the annually aggregated growth rates and SRRs, see Panel B of Table VIII. The results in this section differ from Section II.C in that parameters are calibrated jointly to asset returns, in addition to SRRs and correlations.

The leverage parameter $\phi^v$ of value stocks is estimated to be 8.17, which is much higher than that of growth stocks $\phi^g = 4.68$. Growth stocks have less exposure to the LRR. The differential exposure to the LRR affects the correlations with the growth rates of the annually aggregated consumptions. In our sample, the correlation between the growth rates of the annually aggregated consumptions and dividends of value stocks $\text{Corr}(\Delta c, \Delta d^v)$ is 0.588, while that between consumption and growth stocks $\text{Corr}(\Delta c, \Delta d^g)$ is 0.323. The higher correlation with value stocks is replicated by the higher loadings on the LRR in the dividend growth rate dynamics.

Our model generates SRRs similar to their empirical levels. We find that most variations consumption growth come from LRR, whereas SRRs account for most variations in dividend growth. Although the volatility of the growth rates of the annually aggregated consumptions is 0.865%, the mean of SRR is only about $\sqrt{0.0283\%} \approx 0.168\%$.

## B. Asset Returns

[Table IX about here.]

See Table IX for the model-implied and sample moments on asset return patterns. The model replicates the dynamics of the risk-free rate closely, matching its mean, volatility and first-order autocorrelation. The model generates realistic levels of the equity risk premium 5.461%, compared with 4.803% in data. The model further replicates excess returns in growth and value stocks, hence generating a significant value premium. The low persistence in stock returns is captured by the model. The model further generates realistic levels of volatilities. The volatilities of annualized excess returns in market, growth and value portfolios in the model match their empirical values closely. Moreover, the model produces realistic price-dividend ratios, where growth stocks have much higher valuation ratio (price-dividend ratio) than value stocks.

[Table X about here.]

Table X decomposes the risk premia in the market portfolio, growth and value stocks. The risk premia arise from different channels. According to Proposition 3, the risk premia can be decomposed into the compensation for the instantaneous covariance between the growth rates between the growth rates of consumption and dividends, the LRR and the SRR. As observed from Table X, both the SRR and the LRR matter for asset returns, and the SRR accounts for most of the risk premia.

A few comments are in order here. First, in the baseline calibration $\gamma = 2.4899$, the risk aversion lies in the reasonable range between one and ten documented in Mehra and Prescott (1985). Moreover, the risk aversion $\gamma$ is smaller than in Bansal and Yaron (2004). The EIS $\psi$ is

$\psi = 1.0325 > \frac{1}{\gamma}$, so that the representative agent has a preference for the early resolving of risk. The EIS is also smaller than in Bansal and Yaron (2004). Consequently, the representative agent requires less compensation for the LRR. For the model to generate realistic levels of the risk premia in the cross section, the compensation for the SRR must be sufficiently large. Second, although the leverage parameter of value stocks $\phi^v$ is larger than growth stocks $\phi^g$, the difference in LRR alone is not sufficient to account for the value premium. Therefore, SRRs in value and growth stocks contribute a significant proportion to the observed value premium.

### *C. Predictability*

We demonstrated that SRR could predict the future market excess returns and value-minus-growth returns in Section II.B. The predictability could be partially explained by the business cycle: SRR in consumption is counter-cyclical, and SRR in growth (value) stocks is pro-cyclical (counter-cyclical). Our model replicates the link between the SRR and asset returns. In the data, the predictability for the market excess returns peaks at the four-quarter horizon, and the predictability for the value-minus-growth returns increases in horizons of up to 12 quarters. Our model focus on those horizons where SRR has the most predictability.

We try to model the correlation structure between the cyclical SRRs documented in Section II.A. The loadings $\delta_c\delta_c' + \chi_c$ of transient consumption volatility on the components corresponding to growth stocks are negative, while those on the components corresponding to value stocks are positive. These loadings mimic the small correlation between SRR in consumption and growth stocks, and the large correlation between SRR in consumption and value stocks. While a univariate volatility structure cannot generate a negative correlation between market risk premium and volatility, our model with dynamic covariance structure resolves the negative correlation. [20] The details for the derivations of the model-implied regression coefficients are given in Appendix B.4.

[Table XI about here.]

[20] Although The negative values on $\chi_c$ raise the potential concern that $\sigma_{c,t}$ could turn negative, the probability for $\sigma_{c,t}$ to reach 0 in our calibration is around 0.00002. In a simulation of 2,400,000 months, $\sigma_{c,t}$ only turned negative in 118 months.

See Table XI for the coefficients of the predictive regressions. Our model implies a negative coefficient using the SRR in consumption to predict the future market excess returns, which lies within one standard error from the sample estimate at four-quarter horizon. The SRR in growth stocks has a positive model-implied coefficient in predicting the market excess returns, albeit smaller than the sample counterpart. The model-implied regression coefficient of the SRR in growth stocks is less than one standard error away from sample estimates, where the coefficients are positive both in-sample and in-model. The model-implied coefficients of the SRR in value stocks is slightly outside the one-standard-deviation interval, but have the same negative sign as in the data. Overall, our model does a good job replicating the forecasting patterns of SRRs for the future market excess returns.

We have shown that to predict the future value-minus-growth returns, the SRR in consumption is the only significant predictor among SRRs. Hence, we focus only on the SRR in consumption for the forecasts. We find that the model-implied coefficient is less than one standard error away from the sample estimate at 12-quarter horizon.

## V. Robustness

In this section, we check the robustness of our results to alternative measurement of cash flows and log-linear approximation. We also empirically investigate the results replacing consumption by industrial production index.

### *A. Dividends adjusted for repurchases*

In this part, we adjust dividends to account for equity repurchases by the method proposed in Bansal et al. (2005a). Details about the adjustment method can be found in Appendix I.

[Figure 4 about here.]

[Table XII about here.]

First, we study the properties of SRRs measured with dividends adjusted for repurchases. From Figure 4, SRRs in adjusted dividends also fluctuate with the business cycle. We run the regression (9) with adjusted cash flows to confirm this, see Table XII. Similar to the case using cash dividends, the SRRs in consumption and value stocks negatively predict the future equity premia at the four-quarter horizon, and the predictability declines as the horizon expands. The adjusted $R^2$ at the four-quarter horizon is 10.5%, which remains reasonably large. The SRR in consumptions positively predicts the future value premia and the predictive power scales up with horizon. Meanwhile, SRRs in growth and value stocks are less significant in predicting the future value premia. In summary, in line with the case of cash dividends, SRRs fluctuate with the business cycle and carry similar signals in predicting future returns.

### B. *Errors in log-linearization*

In this part, we quantify the impact of approximation via the log-linearization. To solve the equilibrium in the LRR model, we adapt the projection method proposed in Pohl, Schmedders, and Wilms (2018). Instead of the affine specification of the four state variables (long-run risk $x_t$ and three free components of $\Sigma_t$) in logarithms of consumption-wealth ratio and dividend-price ratio, we assume that $\log \frac{C_t}{W_t}$ and $\log \frac{D^i}{P^i}$ are the sum of products of Chebyshev polynomials of state variables. We set the interval of state variables a bit larger than would have been realized in Monte-Carlo simulations. The degree of Chebyshev polynomials is set to five. Pohl, Schmedders, and Wilms (2018) suggests that the projection method is sufficiently accurate to solve the general equilibrium numerically.

[Table XIII about here.]

Table XIII lists the simulated sample moments for the log-linearized solutions and the projection method based solutions, under the baseline calibration of parameters. Because Table XIII is based on simulation, the results obtained through log-linearization could differ from theoretical moments. Compared with the more accurate solution by the projection method, the log-linearized solution provides moments with economically negligible errors, while providing better tactability by admitting quasi-closed-form solutions.

Pohl, Schmedders, and Wilms (2018) study several models in the LRR framework. They find that the log-linearization approximation induces large errors, especially when the risk aversion $\gamma$ and the EIS $\phi$ are large, or the LRR component $X_t$ is highly persistent. Although the LRR component remains persistent in our study, our model implies a calibration with a small risk aversion $\gamma$ around three and an EIS close to one, which are both smaller than in models studied in Pohl, Schmedders, and Wilms (2018). Therefore, solving our model with log-linearization approximation induces less errors than those analyzed in Pohl, Schmedders, and Wilms (2018), and the approximated solution suffices for our analysis.

### C. *Results using the industrial production index*

In this part, we look into the short-run industrial product index risk (henceforth SRIR), which are defined similarly to SRR but with the production index replacing the role of consumption. We estimate SRIRs empirically, and we then run the predictive regressions of future returns on SRIRs. Figure 5 resembles Figure 1 to plot SRIRs. Table XIV shows the results regarding SRIR in a similar manner to Table XII.

[Figure 5 about here.]

[Table XIV about here.]

The industrial production index growth in Panel B of Figure 5 appears to be less volatile than consumption growth most of the time, except during the Great Recession and the Oil Crisis. In Panel A of Figure 5 it can be seen that SRIRs in cash flows are stable except around the Great Recession. Other than these periods, variation in industrial production is disconnected from economic outlook. The variation in industrial production index growth is less informative about the business cycle than the variation in consumption growth.

In Table XIV, predictive regressions SRIRs explain little variation in future market excess returns and value-minus-growth returns. The discrepancy in predictive power between consumption and industrial production is unlikely to be merely due to measurement errors in monthly consumption data. It is also unlikely that the pure noises in monthly consumption predict future returns.

A more plausible explanation for the discrepancy involves investigating the economic differences in the composition of industrial production index and consumption. We leave this aspect to further studies.

## VI. Conclusions

This paper studies the relationship between SRRs in consumption, Book-to-Market portfolios, the business cycle and asset returns. The SRRs vary with the business cycle. The SRR in growth stocks predicts the future equity premia negatively, while the SRRs in growth stocks and consumption predict the future equity premia positively. For the future one-year (three-year) horizon excess market (value-minus-growth) returns, the adjusted $R^2$ of forecasting regression is 17.6% (11.5%). This predictability remains robust in out-of-sample regressions.

To capture the cyclical variations in SRR and asset returns, we propose a LRR model where the stochastic covariance process is modeled by a Wishart process. The model reproduces the growth dynamics in consumption and dividends, the cross-sectional asset pricing moments (particularly the value premium), and the coefficients of the predictive regressions. Both SRR and LRR components contribute to the equity and value risk premia.

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# Appendix

## Appendix A. Wishart process

In this appendix, we summarize some essential properties of the Wishart process for solving our model. More details can be found, for example, in Gourieroux et al. (2009).

### *A.1. Moments and autocovariances*

In this section, we will give the first two moments of the Wishart process without detailed derivations. For $l \in \mathbb{R}^+$, denote $A_l := \exp(lM)$ and $\Xi_l := \int_0^l \exp(sM)QQ'[\exp(sM)]'ds$. To calculate matrix exponential, we use the formula in Van Loan (1978). Gourieroux et al. (2009) gives

$$\mathbb{E}_t(\Sigma_{t+l}) = A_l \Sigma_t (A_l)' + K\Xi_l \tag{A1}$$

Let $\Sigma(\infty) = \mathbb{E}(\Sigma_t)$ denote the expectation of the Wishart process in the stationary distribution. Let $\Xi(\infty) = \Sigma(\infty)/K$ be the counterpart of $\Xi$ in the stationary distribution, then $\mathbb{E}(\Sigma_t) = \Sigma(\infty) = K\Xi(\infty)$. $\Sigma(\infty)$ is solved by

$$\Sigma(\infty) = A_l \Sigma(\infty) A_l' + K\Xi_l \tag{A2}$$

Let $h_1$, $h_2$ be $n \times n$ constant symmetric matrices, $l > 0$, the second moments of the Wishart process are given as

$$\begin{aligned}
\mathrm{Cov}(\mathrm{Tr}(h_1\Sigma_t), \mathrm{Tr}(h_2\Sigma_t)) &= 2K\,\mathrm{Tr}(\Xi(\infty)h_1\Xi(\infty)h_2) \\
\mathrm{Cov}[\mathrm{Tr}(h_1\Sigma_{t+l}), \mathrm{Tr}(h_2\Sigma_t)] &= \mathrm{Cov}[\mathbb{E}_t(\mathrm{Tr}(h_1\Sigma_{t+l})), \mathrm{Tr}(h_2\Sigma_t)] \\
&= \mathrm{Cov}[\mathrm{Tr}((\exp(lM))'h_1\exp(lM)\Sigma_t), \mathrm{Tr}(h_2\Sigma_t)] \\
&= 2K\,\mathrm{Tr}(\Xi(\infty)(\exp(lM))'h_1\exp(lM)\Xi(\infty)h_2)
\end{aligned} \tag{A3}$$

To solve the model, we also need calculate the second moments of the integrated Wishart process

$$\mathrm{Var}\left(\mathrm{Tr}(h_1\int_t^{t+1}\Sigma_s ds)\right),\ \mathrm{Cov}\left(\mathrm{Tr}(h_1\int_t^{t+1}\Sigma_s ds), \mathrm{Tr}(h_2\int_{t-1}^{t}\Sigma_u du)\right)$$

Equation (A3) reduces this problem to the calculation of matrix exponential integrals, for which there exists closed-form solutions (see Van Loan (1978)).

### *A.2. Quadratic variation of matrix SDE*

Here, we study the quadratic variation of traces of matrix stochastic processes. Denote the quadratic variation by $\langle , \rangle$.

Lemma A.1: *Assume $W_t$ is a $n \times n$ matrix Brownian Motion, and $A_t, \bar{A}_t$ are predictable $n \times n$ matrix processes. Then $\langle \mathrm{Tr}(\int A_. dW_.), \mathrm{Tr}(\int \bar{A}_. dW_.)\rangle_t = \mathrm{Tr}(A_t\bar{A}_t')$.*

*Proof.*

$$\begin{aligned}
\langle \mathrm{Tr}(\int A_. dW_.), \mathrm{Tr}(\int \bar{A}_. dW_.)\rangle_t &= \langle \sum_{i,j=1,\dots,n}\int A_t^{ij}dW_t^{ji}, \sum_{l,m=1,\dots,n}\int \bar{A}_t^{lm}dW_t^{ml}\rangle_t \\
&= \sum_{i,j=1,\dots,n} A_t^{ij}\bar{A}_t^{ji} \\
&= \mathrm{Tr}(A_t\bar{A}_t')
\end{aligned}$$

$\square$

Given that we work with Wishart process, the following corollary comes in handy.

Corollary A.2: *Assume $\Sigma_t$ is the Wishart process given in* (18). *Let $A_t, \bar{A}_t$ be predictable $n \times n$ symmetric matrix processes. Then $\langle \mathrm{Tr}(\int A_. d\Sigma_.),\ \mathrm{Tr}(\int \bar{A}_. d\Sigma_.)\rangle_t = \mathrm{Tr}(4QA_t\Sigma_t\bar{A}_tQ)$.*

# Appendix B. Details of the model solutions

## *B.1. Proof of proposition 1*

Given the affine structure of the underlying problem, we guess the following exponential affine form for the value function:

$$J(W_t, X_t, \Sigma_t) = \exp(A_0 + A_1 X_t + \mathrm{Tr}(A_2 \Sigma_t)) \frac{W_t^{1-\gamma}}{1-\gamma}. \tag{B1}$$

Because $\Sigma_t$ is symmetric, w.l.o.g. we can assume $A_2$ to be a symmetric matrix. From the optimization problem in (27), we obtain the Hamilton-Jacobi-Bellman (HJB) equation as:

$$\max_C \{f(C, J) + \mathcal{A}^c J\} = 0, \tag{B2}$$

where $\mathcal{A}^c$ is the infinitesimal generator associated with state variables $(W_t, X_t, \Sigma_t)$. The first order condition of the HJB equation for consumption $C_t$ is

$$\frac{1-\gamma}{W} J = J_W = f_C = \beta(1-\gamma) \frac{C^{-1/\psi} J}{((1-\gamma)J)^{\frac{1-1/\psi}{1-\gamma}}}. \tag{B3}$$

For notational convenience, we define

$$G_t := A_0 + A_1 X_t + \mathrm{Tr}(A_2 \Sigma_t). \tag{B4}$$

Then, from the first-order condition (B3), we obtain the consumption-wealth ratio $\frac{C_t}{W_t}$ as:

$$\frac{C_t}{W_t} = \beta^{\psi} \exp(\frac{1-\psi}{1-\gamma} G_t) = \beta^{\psi} \exp(A_{0a} + A_{1a} X_t + \mathrm{Tr}(A_{2a} \Sigma_t)). \tag{B5}$$

where $A_{ka} = \frac{1-\psi}{1-\gamma} A_k$, for $k = 0, 1, 2$. The consumption-wealth ratio is an exponential affine function of the state variables. Note that if $\psi = 1$, consumption-wealth ratio $\frac{C_t}{W_t}$ is constant and equal to $\beta$. By substituting (B5) back into (B1), we get

$$J_t(C_t, X_t, \Sigma_t) = \exp(\psi G_t) \beta^{-\psi(1-\gamma)} \frac{C_t^{1-\gamma}}{1-\gamma} \tag{B6}$$

Then, for the case $\psi \neq 1$, the HJB equation can be rewritten as:

$$\begin{aligned} 0 &= \beta\theta[\beta^{\psi-1}\exp(\frac{1-\psi}{1-\gamma}G_t)-1]+\psi\frac{\mathbb{E}_t[dG_t]}{dt}+\frac{\psi^2}{2}\frac{(dG_t)^2}{dt} \\ &\quad +(1-\gamma)\frac{\mathbb{E}_t[dC_t]}{C_t dt}+(1-\gamma)(-\gamma)\frac{(dC_t)^2}{2C_t^2 dt}+(1-\gamma)\psi\frac{dC_t dG_t}{C_t dt} \qquad \text{(B7)} \\ &= \theta(\frac{C_t}{W_t}-\beta)+\psi\left(-\alpha A_1 X_t+\mathrm{Tr}[A_2(\Omega\Omega'+M\Sigma_t+\Sigma_t M')]\right) \\ &\quad +\frac{\psi^2}{2}(A_1^2\delta_x'\Sigma_t\delta_x+\mathrm{Tr}(4QA_2\Sigma_t A_2 Q)) \\ &\quad +(1-\gamma)(\mu_c+X_t)+\frac{(1-\gamma)(-\gamma)}{2}(\delta_c'\Sigma_t\delta_c+\mathrm{Tr}(\chi_c\Sigma_t)+\bar{\sigma}_c^2) \qquad \text{(B8)} \end{aligned}$$

and, for $\psi = 1$,

$$\begin{aligned} 0 &= \beta(1-\gamma)(\log(\beta)-\frac{G_t}{1-\gamma})+\psi\left(-\alpha A_1 X_t+\mathrm{Tr}[A_2(\Omega\Omega'+M\Sigma_t+\Sigma_t M')]\right) \\ &\quad +\frac{1}{2}(A_1^2\delta_x'\Sigma_t\delta_x+\mathrm{Tr}(4QA_2\Sigma_t A_2 Q)) \\ &\quad +(1-\gamma)(\mu_c+X_t)+\frac{(1-\gamma)(-\gamma)}{2}(\delta_c'\Sigma_t\delta_c+\mathrm{Tr}(\chi_c\Sigma_t)+\bar{\sigma}_c^2). \qquad \text{(B9)} \end{aligned}$$

To obtain the coefficients of the representation in (B1) for the case when $\phi \neq 1$, we adopt the standard log-linear approximation of Campbell and Shiller (1988). Defining $c_t := \log C_t$ and $w_t := \log W_t$, we approximate the consumption-wealth ratio as

$$\frac{C_t}{W_t} = \exp(c_t - w_t) \approx g_1 - g_1 \log g_1 + g_1(c_t - w_t), \qquad \text{(B10)}$$

where $g_1 = \exp(\mathbb{E}(c_t - w_t))$. Because $C_t/W_t$ depends on $A_0$, $A_1$, $A_2$, which in turn depend on $g_1$, it is not possible to give an analytical expression of $g_1$. Hence, $g_1$ must be calculated numerically. See Appendix B.5 for details. Then, the log-linearized HJB equation is

$$\begin{aligned} 0 &= \theta[g_1 - g_1\log g_1 + g_1(\psi\log\beta + A_{0a} + A_{1a}X_t + \mathrm{Tr}(A_{2a}\Sigma_t)) - \beta] \\ &\quad + \psi\left(-\alpha A_1 X_t+\mathrm{Tr}[A_2(\Omega\Omega'+M\Sigma_t+\Sigma_t M')]\right)+(1-\gamma)(\mu_c+X_t) \\ &\quad + \frac{\psi^2}{2}(A_1^2\delta_x'\Sigma_t\delta_x+4QA_2\Sigma_t A_2 Q)-\frac{(1-\gamma)\gamma}{2}(\delta_c'\Sigma_t\delta_c+\mathrm{Tr}(\chi_c\Sigma_t)+\bar{\sigma}_c^2). \qquad \text{(B11)} \end{aligned}$$

For the case of $\psi = 1$, no approximation is needed since the log-linearization is exact. The resulting HJB equation is

$$\begin{aligned} 0 &= \beta(1-\gamma)\log(\beta) - \beta(A_0 + A_1 X_t + \mathrm{Tr}(A_2\Sigma_t)) \\ &\quad - \alpha A_1 X_t + \mathrm{Tr}[A_2(\Omega\Omega' + M\Sigma_t + \Sigma_t M')] + (1-\gamma)(\mu_c + X_t) \\ &\quad + \frac{1}{2}(A_1^2 \delta_x' \Sigma_t \delta_x + 4QA_2\Sigma_t A_2 Q) - \frac{(1-\gamma)\gamma}{2}(\delta_c'\Sigma_t\delta_c + \mathrm{Tr}(\chi_c\Sigma_t) + \bar{\sigma}_c^2). \end{aligned} \tag{B12}$$

Now we solve for $A_0$, $A_1$, and $A_2$. Irrespective of the value of $\psi$, $A_1$ satisfies

$$-g_1\psi A_1 - \alpha\psi A_1 + (1-\gamma) = 0, \tag{B13}$$

If $\psi = 1$, then $g_1 = \beta$. For $\psi > 1$, $A_0$ satisfies

$$\theta(g_1 - g_1 \log g_1 + g_1\psi \log \beta) - \beta\theta - g_1\psi A_0 + \psi\,\mathrm{Tr}(A_2\Omega\Omega') + (1-\gamma)\mu_c + \frac{(1-\gamma)(-\gamma)}{2}\bar{\sigma}_c^2 = 0. \tag{B14}$$

For $\psi = 1$, we have

$$\beta(1-\gamma)\log(\beta) - \beta A_0 + \mathrm{Tr}(A_2\Omega\Omega') + (1-\gamma)\mu_c + \frac{(1-\gamma)(-\gamma)}{2}\bar{\sigma}_c^2 = 0. \tag{B15}$$

To obtain $A_2$, we first note that the terms involving $\Sigma_t$ in the HJB equation (B11) should sum up to zero:

$$\mathrm{Tr}\left[\left(-g_1\psi A_2 + \psi(M'A_2 + A_2M) + \frac{\psi^2}{2}(A_1^2\delta_x\delta_x' + 4A_2QQA_2) + \frac{(1-\gamma)(-\gamma)}{2}(\delta_c\delta_c' + \chi_c)\right)\Sigma_t\right] = 0$$

If we denote the matrix left-multiplying $\Sigma_t$ inside the trace operator by $L$, then $L$ must satisfy $L + L' = 0$ because $\Sigma_t$ is symmetric. $L$ does not have to be a zero matrix. Thus,

$$A_2 g_1\psi - \psi(A_2M + M'A_2) = \frac{\psi^2}{2}(A_1^2\delta_x\delta_x' + 4A_2QQA_2) + \frac{(1-\gamma)(-\gamma)}{2}(\delta_c\delta_c' + \chi_c) \tag{B16}$$

We then solve for a symmetric $A_2$ numerically from (B16). This equation has the form of a generalized continuous time algebraic Riccati equation, which have a positive semidefinite solution under certain assumptions.[21] In particular, the generalized continuous time algebraic Riccati equation for $X$ is of the form

$$A'XE + E'XA - (E'XB + S)R^{-1}(B'XE + S') + V = 0, \tag{B17}$$

[21] See Kawamoto et al. (1999) and Bittanti et al. (2012).

where $A$, $Q$ and $E$ are square matrices of the same dimension. Furthermore, $Q$ and $R$ are symmetric matrices. Hence, in our case,

$$BR^{-1}B' = 2\psi^2 QQ,$$
$$SR^{-1}S' - V = \frac{\psi^2}{2}A_1^2\delta_x\delta_x' - \frac{(1-\gamma)\gamma}{2}(\delta_c\delta_c' + \chi_c),$$
$$A - BS' = \frac{g_1\psi}{2}I - \psi M.$$

## B.2. Proof of proposition 2

To derive the state price deflator, we take partial derivatives of $f(C,J)$ and use identities (B5) and (B6) to obtain:

$$f_J(C_t,J_t) = \begin{cases} (\theta-1)\frac{C_t}{W_t} - \beta\theta & \text{if } \psi \neq 1, \\ \beta(1-\gamma)[\log\beta - \frac{G_t}{1-\gamma}] - \beta & \text{if } \psi = 1, \end{cases}$$
$$f_C(C_t,J_t) = \frac{\beta^{\psi\gamma}\exp(\frac{1-\psi\gamma}{1-\gamma}G_t)C_t^{-\gamma}}{1-\gamma}.$$

The expression for SDF under recursive utility is, according to Duffie and Epstein (1992),

$$\pi_t = \exp[\int_0^t f_J(C_s,J_s)ds]f_C(C_t,J_t), \tag{B18}$$

By plugging in expressions of $f_J(C_t,J_t)$ and $f_C(C_t,J_t)$, we obtain the dynamics of $\pi_t$,

$$\frac{d\pi_t}{\pi_t} = f_J(C_t,J_t)dt + \frac{df_C(C_t,J_t)}{f_C(C_t,J_t)} \tag{B19}$$
$$= -(r_f dt + \mathrm{Tr}(\Lambda^\sigma dB_t^\sigma) + \Lambda dB_t + \Lambda^X dB_t^X + \Lambda^c dB_t^c). \tag{B20}$$

where $\Lambda^\sigma$, $\Lambda$, $\Lambda^X$ and $\Lambda^c$ are the prices of risk, which we can identify as

$$\Lambda^\sigma = -2(\frac{1-\psi\gamma}{1-\gamma})QA_2\sqrt{\Sigma_t} \tag{B21}$$
$$\Lambda = \gamma\delta_c'\sqrt{\Sigma_t} \tag{B22}$$
$$\Lambda^X = -(\frac{1-\psi\gamma}{1-\gamma})A_1\sqrt{\delta_x'\Sigma_t\delta_x}, \tag{B23}$$
$$\Lambda^c = \gamma\sigma_c. \tag{B24}$$

We can read off risk-free interest rate directly from SDF. The risk-free interest rate can be decomposed in the following:

$$r_f = r_0 + r_x X_t + \mathrm{Tr}(r_\Sigma \Sigma_t), \tag{B25}$$

where by matching constants and coefficients on $X_t$ and $\Sigma_t$, we obtain

$$r_0 = \begin{cases} -(\theta-1)(g_1 - g_1\log(g_1) + g_1\psi\log(\beta) - \frac{g_1\psi}{\theta}A_0) \\ +\beta\theta - \frac{1-\psi\gamma}{1-\gamma}\mathrm{Tr}(A_2KQQ) + \gamma\mu_c - \gamma(\gamma+1)\frac{\sigma_c^2}{2} & \text{if } \psi > 1 \\ -\beta(1-\gamma)(\log(\beta) - A_0/(1-\gamma)) + \beta \\ -\mathrm{Tr}(A_2KQQ) + \gamma\mu_c - \gamma(\gamma+1)\bar{\sigma}_c^2/2 & \text{if } \psi = 1 \end{cases} \tag{B26}$$

$$\begin{aligned} r_\Sigma &= g_1\left(\frac{1-\psi\gamma}{1-\gamma}\right)A_2 - \left(\frac{1-\psi\gamma}{1-\gamma}\right)(A_2M + M'A_2) \\ &\quad -\frac{1}{2}\left(\frac{1-\psi\gamma}{1-\gamma}\right)^2(A_1^2\delta_x\delta_x' + 4A_2QQA_2) - \frac{\gamma(\gamma+1)}{2}(\delta_c\delta_c' + \chi_c) \end{aligned} \tag{B27}$$

$$r_x = \frac{1}{\psi}. \tag{B28}$$

### B.3. Proof of proposition 3

Assume that the dividend-price ratio has the following exponential affine form,

$$\frac{D_t^i}{P_t} = \exp(A_{0i} + A_{1i}X_t + \mathrm{Tr}(A_{2i}\Sigma_t)), \tag{B29}$$

where $A_{2i}$ is a symmetric matrix. The instantaneous return of asset $i$ is

$$\begin{aligned} \frac{dP_t^i}{P_t^i} &= (\mu_i + \phi_i X_t)dt + \delta_i'\sqrt{\Sigma_t}dB_t + \sigma_i dB_t^i \\ &\quad -[A_{1i}(-\alpha X_t dt + \sqrt{\delta_x'\Sigma_t\delta_x}dB_t^X)] - \mathrm{Tr}[A_{2i}(\Omega\Omega' + M\Sigma_t + \Sigma_t M')dt + 2QA_{2i}\sqrt{\Sigma_t}dB_t^\sigma] \\ &\quad +\frac{1}{2}A_{1i}^2\delta_x'\Sigma_t\delta_x dt + 2\,\mathrm{Tr}(QA_{2i}\Sigma_t A_{2i}Q)dt. \end{aligned} \tag{B30}$$

We perform a log-linear approximation as in Campbell and Shiller (1988). Defining $d_t^i = \log D_t^i$ and $p_t^i = \log P_t^i$, then

$$\frac{D_t^i}{P_t^i} \approx g_{0i} + g_{1i}(A_{0i} + A_{1i}X_t + \mathrm{Tr}(A_{2i}\Sigma_t)), \tag{B31}$$

where $g_{1i} = \exp(\mathbb{E}(d_t^i - p_t^i))$ and $g_{0i} = g_{1i} - g_{1i}\log g_{1i}$. Because $d(\pi_t P_t) + \pi_t D_t dt$ must have zero drift,

$$\mathbb{E}_t\left[\frac{dP_t}{P_t}\right] + \frac{D_t}{P_t}dt = r_f dt - \frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}, \tag{B32}$$

where $-\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}$ is the risk premium of the asset. The formula for the risk premium is the following:

$$-\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}/dt = \gamma\delta_c'\Sigma_t\delta_i + \psi(1-\frac{1}{\theta})A_1A_{1i}\delta_x'\Sigma_t\delta_x + 4\psi(1-\frac{1}{\theta})\operatorname{Tr}(QA_{2i}\Sigma_tA_2Q) \tag{B33}$$

where we used Proposition A.1 to calculate the quadratic variation of Wishart diffusions. By comparing coefficients in Equations (B30) and (B32), we find that $A_{0i}$ must satisfy

$$\mu_i - \operatorname{Tr}(A_{2i}\Omega\Omega') + g_{0i} + g_{1i}A_{0i} = r_0. \tag{B34}$$

Similarly, for $A_{1i}$:

$$\phi_i + \alpha A_{1i} + g_{1i}A_{1i} = r_x, \tag{B35}$$

and for $A_{2i}$:

$$\begin{aligned}&\operatorname{Tr}[(-2A_{2i}M + \frac{A_{1i}^2}{2}\delta_x\delta_x' + 2A_{2i}QQA_{2i} + g_{1i}A_{2i})\Sigma_t]\\&= \operatorname{Tr}[(r_\Sigma + \gamma\delta_i\delta_c' + 4(\frac{1-\psi\gamma}{1-\gamma})A_2QQA_{2i} + (\frac{1-\psi\gamma}{1-\gamma})A_1A_{1i}\delta_x\delta_x')\Sigma_t].\end{aligned} \tag{B36}$$

Hence, $A_{2i}$ is a solution to:

$$\begin{aligned}&-(A_{2i}M + M'A_{2i}) + \frac{A_{1i}^2}{2}\delta_x\delta_x' + 2A_{2i}QQA_{2i} + g_{1i}A_{2i}\\&= r_\Sigma + \gamma\frac{\delta_d\delta_c' + \delta_c\delta_d'}{2} + (\frac{1-\psi\gamma}{1-\gamma})\left(2A_2QQA_{2i} + 2A_{2i}QQA_2 + A_1A_{1i}\delta_x\delta_x'\right).\end{aligned} \tag{B37}$$

To obtain $A_{2i}$ numerically, we can again cast it into the form of a generalized continuous time algebraic Riccati equation (B17). In this case,

$$\begin{aligned}SR^{-1}S' - V &= \frac{A_{1i}^2}{2}\delta_x\delta_x' - r_\Sigma - \gamma\frac{\delta_d\delta_c' + \delta_c\delta_d'}{2} - (\frac{1-\psi\gamma}{1-\gamma})A_1A_{1i}\delta_x\delta_x',\\A - BS' &= -\frac{g_{1i}}{2}I + M + 2(\frac{1-\psi\gamma}{1-\gamma})QQA_2,\\B &= \sqrt{2}Q.\end{aligned}$$

Given proper technical conditions, a positive semidefinite solution $A_{2i}$ exists.

### *B.4. Proof of proposition 4*

In this subsection, we derive model-implied regression coefficients, in which we regress future asset return on SRRs.

We want to study the regressions of the form

$$r^e_{t,t+\frac{Q}{4}} = \beta_0 + \beta_1 SRR^c_{t-1,t} + \beta_2 SRR^g_{t-1,t} + \beta_3 SRR^v_{t-1,t} + \epsilon_{t,t+\frac{Q}{4}} \tag{B38}$$

where

$$\begin{aligned} SRR^c_{t-1,t} &= \int_{t-1}^{t} \delta_c' \Sigma_s \delta_c ds + \int_{t-1}^{t} \mathrm{Tr}(\chi_c \Sigma_t) + \bar{\sigma}_c^2 \\ SRR^i_{t-1,t} &= \int_{t-1}^{t} \delta_c' \Sigma_s \delta_i ds \end{aligned}$$

Thus, we derive the model-implied coefficients for the following regression:

$$r^e_{t,t+\frac{Q}{4}} = \beta_0 + \mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)\beta_1 + \mathrm{Tr}(\int_{t-1}^{t} h_2 \Sigma_s ds)\beta_2 + \cdots + \mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)\beta_m + \epsilon_{t,t+\frac{Q}{4}}$$

where $r^e$ is excess stock return over risk-free rate in the period from $t$ to $t + \frac{Q}{4}$, $h_i \in \mathbb{R}^{3\times 3}$ for $i = 1, 2, ..m$. Denote by $\boldsymbol{\beta} := (\beta_0, \beta_1, \cdots, \beta_m)$ the vector of model-implied regression coefficients.

For convenience, denote the right-hand-side independent variables by

$$\mathrm{RHSVAR} := (1, \mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds), \cdots, \mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds))'$$

Regression coefficients are therefore

$$\boldsymbol{\beta} = E(\mathrm{RHSVAR} \times \mathrm{RHSVAR}')^{-1} E(\mathrm{RHSVAR} \times r^e_{t,t+\frac{Q}{4}}) \tag{B39}$$

where $E(\mathrm{RHSVAR} \times \mathrm{RHSVAR}')$ is

$$\begin{bmatrix} 1 & \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)] & \cdots & \mathbb{E}[\int_{t-1}^{t} h_m \Sigma_s ds)] \\ \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)] & \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)\, \mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)] & \cdots & \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)\, \mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)] & \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)\, \mathrm{Tr}(\int_{t-1}^{t} h_1 \Sigma_s ds)] & \cdots & \mathbb{E}[\mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)\, \mathrm{Tr}(\int_{t-1}^{t} h_m \Sigma_s ds)] \end{bmatrix}$$

Because the expression of $r^e_{t,t+\frac{Q}{4}}$ is given by Equations (B48)-(B51) and Brownian motions in $r^e_{t,t+\frac{Q}{4}}$ are uncorrelated with $\Sigma_t$, we can calculate $\mathbb{E}(\mathrm{RHSVAR} \times r^e_{t,t+\frac{Q}{4}})$ similarly. Then we use Equation (A3) and techniques in Van Loan (1978) to calculate model-implied $\boldsymbol{\beta}$.

### B.5. Numerically solving $g_1$ and $g_{1i}$

Here we develop the algorithm to calculate $g_1$ and $g_{1i}$ numerically. Recall Equation (30),

$$g_1 = \mathbb{E}(\frac{C_t}{W_t}) = \beta^{\psi} \exp(A_{0a}) \mathbb{E}[\exp(A_{1a} X_t + \mathrm{Tr}(A_{2a}\Sigma_t))] \tag{B40}$$

Then,

$$\begin{aligned} g_1 &= \beta^{\psi} \exp(A_{0a}) \mathbb{E}\left[\mathbb{E}[\exp(A_{1a} X_t + \mathrm{Tr}(A_{2a}\Sigma_t)) | \Sigma_{0\le s\le t}]\right] && \text{(B41)} \\ &= \beta^{\psi} \exp(A_{0a}) \mathbb{E}[\exp(A_{1a} e^{-\alpha t} X_0 + \frac{1}{2} A_{1a}^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} \delta_x' \Sigma_s \delta_x ds + \mathrm{Tr}(A_{2a}\Sigma_t))] && \text{(B42)} \end{aligned}$$

Note that this equation holds for $\forall t > 0$. Obviously $e^{-\alpha t} X_0$ converges to zero in probability as $t \to \infty$, so

$$g_1 = \lim_{t\to\infty} \beta^{\psi} \exp(A_{0a}) \mathbb{E}[\frac{1}{2} A_{1a}^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} \delta_x' \Sigma_s \delta_x ds + \mathrm{Tr}(A_{2a}\Sigma_t))]. \tag{B43}$$

Because $\delta_x' \Sigma_t \delta_x$ has the long term mean $\delta_x' \Sigma(\infty) \delta_x$, we approximate $g_1$ by

$$\begin{aligned} g_1 &\approx \lim_{t\to\infty} \beta^{\psi} \exp(A_{0a}) \mathbb{E}[\frac{1}{2} A_{1a}^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} \delta_x' \Sigma(\infty) \delta_x ds + \mathrm{Tr}(A_{2a}\Sigma_t))] \\ &= \beta^{\psi} \exp(A_{0a} + \frac{1}{4\alpha} A_{1a}^2 \delta_x' \Sigma(\infty) \delta_x) \mathbb{E}[\exp(\mathrm{Tr}(A_{2a}\Sigma_t))]. \end{aligned}$$

Laplace transform of $W(K, 0, \Xi(\infty))$ is given in Gourieroux et al. (2009):

$$\mathbb{E}[\exp(\Gamma \Sigma_t)] = \det(I_n - 2\Xi(\infty)\Gamma)^{-K/2}$$

Thus, we numerically solve for $g_1$ from

$$g_1 = \beta^{\psi} \exp(A_{0a} + \frac{1}{4\alpha} A_{1a}^2 \delta_x' \Sigma(\infty) \delta_x) \det(I_n - 2\Xi(\infty) A_{2a})^{-K/2} \tag{B44}$$

Similarly for asset $i$, its stationary mean of dividend-price ratio $g_{1i}$ is solved from

$$g_{1i} = \exp(A_{0i} + \frac{1}{4\alpha} A_{1i}^2 \delta_x' \Sigma(\infty) \delta_x) \det(I_n - 2\Xi(\infty) A_{2i})^{-K/2} \tag{B45}$$

### B.6. Theoretical moments

This section gives the analytical expressions for moments used in GMM estimation. There are 36 moments in total. Under our assumption $M$ is negative definite and lower triangular, $Q = qI_n$ and $\delta_x = \eta\delta_c$, the expressions of the following moments can be further simplified. The parameters are:

$$\alpha, \sigma_X, \mu_c, \delta_c, \mu_i, \delta_i, \phi_i, m, q, \beta, \psi, \gamma$$

where $i = m, 1, 2, 3$ represents market portfolio and 3 Fama-French portfolios respectively. There are 26 parameters to be estimated. To make the estimations easier, we restrict $\mu_i$ to be the sample mean of the corresponding mean of cash flow growth, which leaves us with 21 parameters to match 31 moments. To estimate the over-identified system, a weight matrix $W$ is necessary, which we specify as a diagonal matrix that adjusts for the magnitudes of moments. We calculate the standard errors of the moments in the following. Bansal et al. (2016) argues decision interval for long-run risk model should be a month. To reflect the more frequent decision making (i.e., more than once per year), we model the agent to make decision dynamically and continuously. Because observations are only available yearly in aggregate, we calculate theoretical moments at yearly aggregations. $\Delta c_{t,t+1} := \int_t^{t+1} \frac{dC_t}{C_t}$, and $\Delta d^i_{t,t+1}$, $r^i_{e,t,t+1}$, $r_{f,t,t+1}$ are similarly defined. Therefore,

$$\begin{aligned}
\Delta c_{t,t+1} &= \mu_c + \int_t^{t+1} X_s ds + \int_t^{t+1} \delta_c' \sqrt{\Sigma_s} dB_s + \int_t^{t+1} \sigma_{c,t} dB_s && \text{(B46)} \\
\Delta d^i_{t,t+1} &= \mu_i + \phi_i \int_t^{t+1} X_s ds + \int_t^{t+1} \delta_i' \sqrt{\Sigma_s} dB_s + \int_t^{t+1} \sigma_i dB^i_s && \text{(B47)} \\
r^i_{e,t,t+1} &= \gamma \delta_c' \int_t^{t+1} \Sigma_s ds \delta_i + \left(\frac{1-\psi\gamma}{1-\gamma}\right) A_1 A_{1i} \delta_x' \int_t^{t+1} \Sigma_s ds \delta_x && \text{(B48)} \\
&\quad + 4\left(\frac{1-\psi\gamma}{1-\gamma}\right) \mathrm{Tr}(Q A_{2i} \int_t^{t+1} \Sigma_s ds A_2 Q) - A_{1i} \int_t^{t+1} \sqrt{\delta_x' \Sigma_s \delta_x} dB^X_s && \text{(B49)} \\
&\quad - \mathrm{Tr}\left( A_{2i} \int_t^{t+1} \sqrt{\Sigma_s} dB^\sigma_s Q + Q' \left(\int_t^{t+1} \sqrt{\Sigma_s} dB^\sigma_s\right)' A_{2i}' \right) && \text{(B50)} \\
&\quad + \delta_i' \int_t^{t+1} \sqrt{\Sigma_s} dB_s + \int_t^{t+1} \sigma_i dB^i_s && \text{(B51)} \\
r_{f,t,t+1} &= r_0 + r_x \int_t^{t+1} X_s ds + \mathrm{Tr}(r_\Sigma \int_t^{t+1} \Sigma_s ds) && \text{(B52)}
\end{aligned}$$

We plug in Equations (A2) and (A3) to calculate moments of Wishart process, expressions for model implied moments are shown in Table XV.

[Table XV about here.]

# Appendix C. Monthly consumption growth

We define consumption as the sum of nondurable goods and services, where the data are from the U.S. Bureau of Economic Analysis (BEA). Per capita annual consumption data range from 1927 to 2017 in real terms.[22] The monthly consumption data are available from January 1959 to December 2017, as the national aggregate and in nominal dollar amounts.[23] To make the consumption data at monthly and yearly frequency consistent, we construct the monthly per capita consumption in real terms. We divide aggregate nominal monthly consumption by population and the personal consumption deflator to get personal consumption in real terms. Given that population is only measured quarterly, we linearly interpolate quarterly population to get a monthly estimate of the level of population. We also linearly interpolate quarterly personal consumption deflator to get a monthly personal consumption deflator, where the quarterly personal consumption deflator is the ratio of nondurable consumption plus services in nominal terms divided by those in chained dollars.

To compare consumption and dividend growth rates, and to estimate their covariance, one should be cautious about the different constructions of seasonal adjusted consumption and dividends. While monthly personal consumption data from BEA are seasonal adjusted by removing the seasonal component,[24] dividends are seasonal adjusted simply by calculating the yearly moving average. Consequently, any macroeconomic shock has an immediate impact on consumption growth, but affects dividend growth data only after several quarters. Therefore, to make consumption and dividends comparable, we calculate seasonal adjusted consumption as its moving average in the last 12 months. We let one unit of time interval correspond to one year and we set $h = 1/12$. We denote the personal real consumption before seasonal adjustment between time $t$ and time $t+h$ by $C_{t,t+h}$, the contemporary seasonal component by $S_{t,t+h}$. Hence, the seasonal adjusted consumption, corresponding to the data from BEA, is given by $C^{SA}_{t,t+h} = C_{t,t+h} - S_{t,t+h}$. We then calculate the

[22]Table 2.4.5, Personal Consumption Expenditures by Type of Product, where per capita annual consumption denoted in chained dollar (US dollar fixed in 2009) is available.

[23]Table 2.8.5, Personal Consumption Expenditures by Major Type of Product, Monthly. BEA provides chained dollar monthly consumption only from 1999, so we choose this table with longer time series of nominal consumption data.

[24]See, `https://www.bea.gov/faq/index.cfm?faq_id=123`, where X13-ARIMA-SEATS is implemented.

moving average of consumption $C^{MA}_{t,t+h}$ between time $t$ and $t+h$:

$$\begin{aligned} C^{MA}_{t,t+h} &= \frac{1}{12}\sum_{i=0}^{11} C^{SA}_{t-ih,t-ih+h} \\ &= \frac{1}{12}\sum_{i=0}^{11}(C_{t-ih,t-ih+h} - S_{t-ih,t-ih+h}) \\ &\approx \frac{1}{12}\sum_{i=0}^{11} C_{t-ih,t-ih+h}. \end{aligned}$$

Note that the approximation in the last step holds as long as the seasonal components derived from X13-ARIMA-SEATS within a year sum up to a small value close to 0. In principle, we would prefer to calculate moving average of consumption directly from an unadjusted time series of monthly consumption data instead of using $C^{SA}$, but no such data is available as of now. We confirm insensitivity to seasonalization by X13-ARIMA-SEATS in Panel A of Figure 6, where we plot the annual consumption growth directly calculated from the ratio of consecutive annually aggregated consumptions, and that calculated from summing up the monthly changes in the 12-month moving average of the monthly aggregated consumptions. For the years where both data are available, the two series are almost identical with a high correlation at about 99.6%.[25] Panel B of Figure 6 plots the fluctuations of (rescaled) monthly consumption growth around their annual mean.

Yearly aggregated consumption $C_{t,t+1}$ is the sum of the monthly consumption within the year $\sum_{i=1}^{12} C^{MA}_{t+(i-1)h,t+ih}$. Then, we construct the monthly consumption growth in a way similar to how we construct the growth of dividend moving average. Log monthly consumption growth $\Delta c_{t,t+h}$ is

$$\Delta c_{t,t+h} = \log \frac{C^{MA}_{t,t+h}}{C^{MA}_{t-h,t}} = \log \frac{C_{t-11h,t-10h} + \cdots + C_{t,t+h}}{C_{t-12h,t-11h} + \cdots + C_{t-h,t}} \tag{C1}$$

This monthly consumption growth is different from, for example, Bansal et al. (2016), in which the monthly consumption growth is $\log \frac{C_{t,t+h}}{C_{t-h,t}}$. In their setup, the annually aggregated consumption growth is not the sum of the monthly consumption growth rates within the year. In contrast, our construction of monthly consumption growth reflects the monthly changes of the

[25] To further confirm moving average is not sensitive to whether data is seasonalized, we seasonalize dividends and perform moving average in the same manner to calculate dividend growth. Indeed, dividend growth with seasonalization are highly correlated with dividend growth from raw data, with correlations above 97%. Seasonalization before moving average does not alter the key results in this paper and therefore these results are not included in this paper due to the limitations of space.

annual consumption, which sum up to the annual consumption growth:

$$\sum_{k=1}^{12} \Delta c_{t+(k-1)h,t+kh} = \sum_{k=1}^{12} \log \frac{C_{t+(k-1)h-11h,t+(k-1)h-10h} + \cdots + C_{t+(k-1)h,t+(k-1)h+h}}{C_{t+(k-1)h-12h,t+(k-1)h-11h} + \cdots + C_{t+(k-1)h-h,t+(k-1)h}}$$
$$= \Delta c_{t,t+1}$$

[Figure 6 about here.]

# Appendix D. Simulation details in Table VI

The consumption growth is generated from the following specification:

$$\log \frac{C_t}{C_{t-1}} = \mu_c + x_t + \sigma_c \epsilon_t \tag{D1}$$

$$x_t = \rho x_{t-1} + \sigma_x e_t \tag{D2}$$

$$\epsilon_t,\, e_t \sim N(0,1), i.i.d. \tag{D3}$$

where the unit of time is one month. We simulate 1000 years of monthly consumption growth rates and use the last 900 years for estimation to minimize the effect of the choice of initial value of long-run risk $x_0$. See the results in Panel B of Table VI. To study the effect of long-run risk, we study different calibration: baseline parameters ($\rho = 0.975, \sigma_x = 0.0237, \sigma_c = 0.032, \mu_c = 0.16$), and three alternatives with $\rho = 0$, $\sigma_x = 0$ and $\mu_c = 0$ respectively.

The baseline parameters are chosen in a way to match the persistence of long-run risk in Bansal and Yaron (2004) and consumption growth dynamics. In the baseline case, we are able to generate similar short- and long-run consumption growth rates variance to real data, and the long-run risk component accounts for most of the variance annually aggregated consumption growth. Moreover, the within-year monthly autocorrelation is much smaller than the monthly autocorrelation estimated using the whole sample, which confirms that the persistent long-run consumption risk does not affect the within-year autocorrelation of the monthly aggregated consumption growth rates as much as in the autocorrelation of the annually aggregated consumption growth rates. For comparison, we shut down LRR channel in other calibrations, and we find that without LRR the dynamics of consumption growth behave distinctly from real consumption data. In the second column we let the $\rho = 0$ so that $x_t$ is just another source of transitory risk. In this case variance of annually aggregated consumption is smaller due to zero persistence in $x_t$, and variance of monthly growth rates is almost identical to the variance of annual growth rates. All measures of autocorrelation are close to zero. In the third column we let $\sigma_x = 0$ so that $x_t = 0$ throughout. Similar to the second column, variance of monthly growth rates accounts for almost all of the variance of the annually aggregated consumption growth rates. All autocorrelation measures are close to zero. The fourth column studies the sensitivity to mean of consumption growth, and we find the mean of consumption growth rates has no impact on the variance or autocorrelation measures.

# Tables

**Table I**. Correlation Matrix

| | C | G | V | M | I/C | $\log\frac{P}{D}$ | *cay* | CP |
|---|---|---|---|---|---|---|---|---|
| C | 1 | 0.030 | 0.165 | 0.182 | 0.430 | −0.238 | −0.275 | 0.019 |
| G | 0.030 | 1 | −0.195 | 0.394 | 0.034 | −0.134 | 0.098 | 0.064 |
| V | 0.165 | −0.195 | 1 | 0.142 | 0.088 | 0.079 | −0.184 | −0.030 |
| M | 0.182 | 0.394 | 0.142 | 1 | 0.062 | 0.016 | −0.045 | −0.006 |
| I/C | 0.430 | 0.034 | 0.088 | 0.062 | 1 | −0.696 | −0.504 | 0.028 |
| $\log\frac{P}{D}$ | −0.238 | −0.134 | 0.079 | 0.016 | −0.696 | 1 | 0.036 | −0.106 |
| *cay* | −0.275 | 0.098 | −0.184 | −0.045 | −0.504 | 0.036 | 1 | 0.320 |
| CP | 0.019 | 0.064 | −0.030 | −0.006 | 0.028 | −0.106 | 0.320 | 1 |

This table presents the correlation of independent variables in predictive regressions. The independent variables include $\text{SRR}^c_{t-1,t}$ (C), $\text{SRR}^g_{t-1,t}$ (G), $\text{SRR}^v_{t-1,t}$ (V), $\text{SRR}^m_{t-1,t}$ (M), the logarithm of price-dividend ratio ($\log\frac{P}{D}$), the ratio of income over consumption (I/C) in Santos and Veronesi (2006), the *cay* in Lettau and Ludvigson (2001a) and the CP factor in Cochrane and Piazzesi (2005).

**Table II**. Forecast 12-Quarter Value-Minus-Growth Returns, 1959–2017

| | Value-Minus-Growth Returns | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| C | 79.010*** | 81.677*** | | | | | | | | 41.011* |
| | (23.668) | (23.986) | | | | | | | | (24.195) |
| G | 0.401 | | 0.322 | | | | | | | 0.803 |
| | (2.038) | | (2.117) | | | | | | | (1.934) |
| V | 0.632 | | | 1.159 | | | | | | 0.239 |
| | (0.780) | | | (0.893) | | | | | | (0.894) |
| M | | | | | −4.609 | | | | | |
| | | | | | (3.985) | | | | | |
| I/C | | | | | | 902.488** | | | | 1,518.327*** |
| | | | | | | (357.931) | | | | (490.810) |
| $\log \frac{P}{D}$ | | | | | | | −2.438 | | | 7.622 |
| | | | | | | | (3.686) | | | (4.720) |
| *cay* | | | | | | | | −105.818** | | 58.134 |
| | | | | | | | | (53.768) | | (62.475) |
| CP | | | | | | | | | 40.296 | 29.893 |
| | | | | | | | | | (54.860) | (41.012) |
| Constant | 1.043 | 1.082 | 3.626*** | 3.499*** | 3.844*** | −882.744** | 12.420 | 3.610*** | 3.273** | −1,516.661*** |
| | (1.474) | (1.419) | (1.284) | (1.205) | (1.287) | (351.814) | (12.989) | (1.162) | (1.539) | (494.843) |
| Adjusted $R^2$ | 0.115 | 0.119 | -0.004 | 0.010 | 0.007 | 0.192 | 0.013 | 0.077 | 0.002 | 0.276 |

*Note:* *p<0.1; **p<0.05; ***p<0.01

This table presents the regression results forecasting the future 12-quarter value-minus-growth returns from 1959 Q1 to 2017 Q4. The independent variables include $\mathrm{SRR}^c_{t-1,t}$ (C), $\mathrm{SRR}^g_{t-1,t}$ (G), $\mathrm{SRR}^v_{t-1,t}$ (V), $\mathrm{SRR}^m_{t-1,t}$ (M), the logarithm of price-dividend ratio ($\log \frac{P}{D}$), the ratio of income over consumption (I/C) and *cay*. Newey-West standard errors with lag 8 are shown in parentheses, based on which significance level is determined. The variance and covariances are represented in squared percent (%%), and returns are represented in percent (%).

**Table III**. Forecast 4-Quarter Market Excess Returns, 1959–2017

| | Market Excess Returns | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| C | −188.191*** | −202.475*** | | | | | | | | −205.189*** |
| | (44.948) | (56.081) | | | | | | | | (48.645) |
| G | 8.752* | | 10.246** | | | | | | | 7.372 |
| | (5.119) | | (4.934) | | | | | | | (5.306) |
| V | −4.280*** | | | −6.265** | | | | | | −3.804*** |
| | (1.608) | | | (3.067) | | | | | | (1.406) |
| M | | | | | 13.712 | | | | | |
| | | | | | (10.083) | | | | | |
| I/C | | | | | | −386.914 | | | | 1, 266.683 |
| | | | | | | (685.024) | | | | (1, 031.209) |
| $\log \frac{P}{D}$ | | | | | | | −2.372 | | | 2.895 |
| | | | | | | | (5.291) | | | (8.035) |
| *cay* | | | | | | | | 245.627*** | | 216.661* |
| | | | | | | | | (95.262) | | (126.374) |
| CP | | | | | | | | | 169.363 | 68.698 |
| | | | | | | | | | (111.884) | (85.197) |
| Constant | 10.238*** | 10.869*** | 3.673* | 5.465*** | 4.051* | 384.567 | 13.136 | 4.965*** | 3.018 | −1, 243.882 |
| | (2.442) | (2.223) | (2.163) | (1.672) | (2.070) | (673.000) | (18.342) | (1.748) | (2.490) | (1, 035.735) |
| Adjusted $R^2$ | 0.176 | 0.121 | 0.025 | 0.064 | 0.012 | 0.002 | -0.002 | 0.074 | 0.016 | 0.218 |

*Note:* *p<0.1; **p<0.05; ***p<0.01

This table presents the regression results forecasting the future 4-quarter equity premia from 1959 Q1 to 2017 Q4, performed on quarterly data. The independent variables include $\mathrm{SRR}^c_{t-1,t}$ (C), $\mathrm{SRR}^g_{t-1,t}$ (G), $\mathrm{SRR}^v_{t-1,t}$ (V), $\mathrm{SRR}^m_{t-1,t}$ (M), the logarithm of price-dividend ratio ($\log \frac{P}{D}$), the ratio of income over consumption (I/C) and *cay*. Newey-West standard errors with lag 8 are shown in parentheses, based on which the significance level is determined. SRRs are represented in squared percent (%%), and returns are represented in percent (%).

**Table IV**. Summaries of Out-of-Sample Regressions and Strategies

| | Market | C | G | V | All |
|---|---|---|---|---|---|
| Monthly Sharpe Ratio | 0.107 | 0.165 | 0.124 | 0.149 | 0.192 |
| $R^2_{OS}$ | | 0.021 | 0.025 | 0.024 | 0.033 |

This table summarizes the out-of-sample $R^2$ of predictive regressions for the future one-year market excess return and the Sharpe ratios of market-timing strategies derived from out-of-sample estimates. For the strategies in the second to fourth columns, the predictors are resp. $\text{SRR}^c_{t-1,t}$ (C), $\text{SRR}^g_{t-1,t}$ (G), $\text{SRR}^v_{t-1,t}$ (V). For the strategy in the fifth column, all these predictors are used. Out-of-sample predictive regressions estimate coefficients on a rolling basis from the past 35 years. The market timing strategies are formed by multiplying the market excess return by the regression prediction, which is updated every year. We rescale the weights ex post to ensure all portfolios have the same volatility in returns.

**Table V**. Joint GMM Estimation of SRR and LRR

| | Estimate | Std. Error | t-value | p-value |
|---|---|---|---|---|
| $\mu_c$ | 0.8652 | 0.2485 | 3.4819 | 0.0002 |
| $\mu_m$ | 2.0102 | 1.2212 | 1.6461 | 0.0499 |
| $\mu_g$ | 2.2951 | 1.4895 | 1.5408 | 0.0617 |
| $\mu_v$ | 3.8589 | 2.6457 | 1.4586 | 0.0723 |
| $\sigma^2_X$ | 1.4337 | 0.3130 | 4.5804 | 0.0000 |
| $\phi_m$ | 1.4297 | 0.9072 | 1.5759 | 0.0575 |
| $\phi_g$ | 0.1409 | 0.9401 | 0.1499 | 0.4404 |
| $\phi_v$ | 4.9514 | 1.9920 | 2.4856 | 0.0065 |
| $\mu_{\text{SRR}^c}$ | 0.0283 | 0.0051 | 5.5956 | 0.0000 |
| $\mu_{\text{SRR}^m}$ | 0.0272 | 0.0260 | 1.0451 | 0.1480 |
| $\mu_{\text{SRR}^g}$ | 0.0544 | 0.0519 | 1.0493 | 0.1470 |
| $\mu_{\text{SRR}^v}$ | 0.2501 | 0.1176 | 2.1271 | 0.0167 |

This table summarizes the GMM estimation results for moment conditions (15), (16) and (17). For each parameter, the p-value is calculated from the one-sided test of the parameter equal to 0 against larger than 0. The covariance matrix is estimated by Newey-West estimator. Growth rates are in percent and SRRs are in squared percent.

**Table VI**. Statistics of the SRRs and Growth Rates

| Panel A: Statistics of SRRs and Growth Rates, 1959–2017 | | | | |
|---|---|---|---|---|
| | C | M | G | V |
| Mean of $\text{SRR}_{t-1,t}$ | 0.028 | 0.027 | 0.054 | 0.250 |
| SE of $\text{SRR}_{t-1,t}$ | (0.030) | (0.162) | (0.323) | (0.715) |
| $\widehat{\text{Cov}}(\cdot, \Delta c_{t-1,t})$ | 1.488 | 2.113 | 0.261 | 7.478 |
| Mean of $\text{ACF}_{t-1,t}(1/12)$ | 0.355 | −0.261 | −0.163 | −0.243 |
| SE of $\text{ACF}_{t-1,t}(1/12)$ | (0.265) | (0.282) | (0.259) | (0.283) |
| Full Sample ACF(1) | 0.479 | 0.286 | 0.078 | 0.304 |
| Full Sample ACF(1/12) | 0.924 | 0.039 | −0.056 | 0.082 |
| **Panel B: Statistics of SRRs and Growth Rates, Simulations** | | | | |
| | Baseline | $\rho = 0$ | $\sigma_x = 0$ | $\mu_c = 0$ |
| Mean of $\text{SRR}^c_{t-1,t}$ | 0.023 | 0.017 | 0.011 | 0.023 |
| SE of $\text{SRR}^c_{t-1,t}$ | (0.014) | (0.007) | (0.005) | (0.014) |
| $\widehat{\text{Var}}(\Delta c_{t-1,t})$ | 1.673 | 0.020 | 0.012 | 1.673 |
| Mean of $\text{ACF}_{t-1,t}(1/12)$ | 0.188 | −0.086 | −0.076 | 0.188 |
| SE of $\text{ACF}_{t-1,t}(1/12)$ | (0.290) | (0.252) | (0.243) | (0.290) |
| Full Sample ACF(1) | 0.836 | −0.002 | −0.032 | 0.836 |
| Full Sample ACF(1/12) | 0.905 | 0.003 | 0.004 | 0.905 |

Panel A summarizes the statistics of the SRRs, consumption growth (C) and the cash-flow growth of market (M), growth (G) and value portfolios (V). The first row reports the averages of SRRs. The second row lists in parentheses the standard errors of the SRRs. The third row reports the unconditional variance of the annually aggregated consumption growth and the covariance between the annually aggregated consumption growth and cash-flow growth. The fourth row reports the average of the yearly observations of the first-order autocorrelations within the year of monthly aggregated growth rates, and the fifth row their standard errors are reported in parentheses. The sixth row reports the first-order autocorrelations of the annual consumption growth and cash-flow growth, and the seventh row first-order autocorrelation of the monthly consumption growth and cash-flow growth, both calculated using the full sample. Panel B lists the same statistics for simulated consumption growth under different dynamics specifications. We simulate monthly consumption growth for 1000 years and use the last 900 years to calculate statistics. The first column displays baseline calibration ($\rho = 0.975, \sigma_x = 0.0237, \sigma_c = 0.032, \mu_c = 0.16$), the remaining columns correspond to cases where $\rho = 0$, $\sigma_x = 0$ and $\mu_c = 0$. All SRRs, variances and covariances are expressed in squared percent (%%).

**Table VII**. Baseline Calibrations

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\mu_c$ | 0.0090 | $\delta_1^m$ | 0.2970 | $\delta_1^x$ | 0.1853 | $\phi^m$ | 5.5759 | $\gamma$ | 2.4899 |
| $\mu_m$ | 0.0282 | $\delta_2^m$ | 0.1472 | $\delta_2^x$ | 0.2001 | $\phi^g$ | 4.6816 | | |
| $\mu_g$ | 0.0320 | $\delta_1^c$ | 0.0904 | $\sigma^m$ | 0.0808 | $\phi^v$ | 8.1676 | | |
| $\mu_v$ | 0.0205 | $\delta_2^c$ | 0.0181 | $\sigma^g$ | 0.0998 | $\beta$ | 0.0210 | | |
| $\alpha$ | 0.0869 | $\bar{\sigma}^c$ | 0.0025 | $\sigma^v$ | 0.0786 | $\psi$ | 1.0325 | | |

$$Q = \begin{bmatrix} 0.0029 & -0.0006 \\ -0.0006 & 0.0004 \end{bmatrix}, \ M = \begin{bmatrix} -0.1625 & -0.0000 \\ 0.1989 & -0.0875 \end{bmatrix}$$

$$\chi_c = \begin{bmatrix} -168.5532 & -8.2767 \\ -8.2767 & -0.5919 \end{bmatrix} \times 10^{-4}$$

This table reports the choice of values in the baseline calibration. All the matrices are of dimensional $2 \times 2$. $\delta_m = (\delta_1^m, \delta_2^m)'$, $\delta_g = (1, 0)'$, $\delta_v = (0, 1)'$.

**Table VIII**. Consumption and Cash-Flow Growth

| Panel A: Means, Standard Deviations and Autocorrelations | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Model | Data | SE | | Model | Data | SE |
| $\mathbb{E}(\Delta c)$ | 0.8989 | 0.8652 | (0.3029) | $\sigma(\Delta d^g)$ | 10.6117 | 9.3667 | (1.6733) |
| $\mathbb{E}(\Delta d^m)$ | 2.8205 | 2.0102 | (0.9796) | $\sigma(\Delta d^v)$ | 11.9888 | 15.3426 | (3.3309) |
| $\mathbb{E}(\Delta d^g)$ | 3.1957 | 2.2951 | (1.3532) | $AC1(\Delta c)$ | 0.8593 | 0.4763 | (0.1357) |
| $\mathbb{E}(\Delta d^v)$ | 2.0487 | 3.8589 | (1.8820) | $AC1(\Delta d^m)$ | 0.1969 | 0.2847 | (0.0846) |
| $\sigma(\Delta c)$ | 0.7806 | 1.2091 | (0.1271) | $AC1(\Delta d^g)$ | 0.1019 | 0.0761 | (0.1738) |
| $\sigma(\Delta d^m)$ | 8.0935 | 6.7134 | (1.3045) | $AC1(\Delta d^v)$ | 0.3501 | 0.3050 | (0.1495) |
| **Panel B: The Correlations of Annual Growth and SRRs** | | | | | | | |
| | Model | Data | SE | | Model | Data | SE |
| $\mathrm{Corr}(\Delta c, \Delta d^m)$ | 0.4402 | 0.2559 | (0.1310) | $\mathbb{E}(\mathrm{SRR}^m)$ | 0.0324 | 0.0272 | (0.0205) |
| $\mathrm{Corr}(\Delta c, \Delta d^g)$ | 0.3233 | 0.0226 | (0.1046) | $\mathbb{E}(\mathrm{SRR}^g)$ | 0.0817 | 0.0544 | (0.0466) |
| $\mathrm{Corr}(\Delta c, \Delta d^v)$ | 0.5880 | 0.3961 | (0.0933) | $\mathbb{E}(\mathrm{SRR}^v)$ | 0.2552 | 0.2501 | (0.1185) |
| $\mathbb{E}(\mathrm{SRR}^c)$ | 0.0547 | 0.0283 | (0.0086) | | | | |

This table reports the model-implied (Model) and sample (Data) moments of the variables of interest, as well as their corresponding standard deviation (SE) in sample. In Panel A, we summarize the mean ($\mathbb{E}(\cdot)$), standard deviation ($\sigma(\cdot)$) and first-order autocorrelation ($AC1(\cdot)$) of the growth rates of the annually aggregated consumption and cash flows in value and growth stocks. In Panel B, we summarize the correlations of the growth rates in the annually aggregated consumptions and dividends, the means of SRRs, and the means of short-run covariances between dividend growth rates. We consider the cash flows in the market portfolio ($m$), growth stocks ($g$) and value stocks ($v$). Standard deviations are constructed by the delta method with NW errors at eight lags. The growth rates are in percent. SRRs, variances and covariances are in squared percent.

**Table IX**. Asset Returns

| | Model | Data | SE | | Model | Data | SE |
|---|---|---|---|---|---|---|---|
| $\mathbb{E}(r_f)$ | 0.6766 | 0.9931 | (0.4603) | $\mathbb{E}(r_e^g)$ | 4.9702 | 4.5216 | (2.3462) |
| $AC1(r_f)$ | 0.7592 | 0.8406 | (0.0828) | $AC1(r_e^g)$ | 0.0355 | -0.0420 | (0.1555) |
| $\sigma(r_f)$ | 1.8632 | 1.6664 | (0.2443) | $\sigma(r_e^g)$ | 15.6401 | 17.7419 | (2.8189) |
| $\mathbb{E}(r_e^m)$ | 5.4609 | 4.8030 | (2.2349) | $\mathbb{E}(P^g/D^g)$ | 53.7977 | 54.5673 | (15.4136) |
| $AC1(r_e^m)$ | 0.0411 | -0.0661 | (0.1548) | $\mathbb{E}(r_e^v)$ | 7.3582 | 7.9921 | (2.8214) |
| $\sigma(r_e^m)$ | 16.0553 | 16.8272 | (2.7533) | $AC1(r_e^v)$ | 0.0555 | -0.1216 | (0.1610) |
| $\mathbb{E}(P^m/D^m)$ | 39.1322 | 39.9934 | (11.3543) | $\sigma(r_e^v)$ | 19.8531 | 18.0551 | (2.6285) |
| | | | | $\mathbb{E}(P^v/D^v)$ | 20.3240 | 35.3418 | (10.3722) |

This table reports the model-implied (Model) and sample (Data) moments of asset price patterns, including the means ($E(r)$), the standard deviations ($\sigma(r)$) and the first-order autocorrelations ($AC1(r)$) of the annually aggregated returns and the mean of price-dividend ratios ($E(P/D)$). The in-sample standard deviations (SE) are also reported. Standard deviations are constructed by the delta method with NW errors at eight lags. The assets under consideration are risk-less asset ($f$), market portfolio ($m$), growth stocks ($g$) and value stocks ($v$). Returns are in percent.

**Table X**. The Decomposition of the Risk Premium

| | $\gamma\,\mathrm{Cov}_t(\Delta c, \Delta d)$ | LRR | SRR |
|---|---|---|---|
| Market | 0.0008 | 0.4752 | 4.9849 |
| Growth | 0.0020 | 0.4082 | 4.5600 |
| Value | 0.0014 | 0.6134 | 6.7434 |

This table reports the decomposition of the risk premium in market portfolio, growth and value stocks. The risk premium can be attributed to three sources: the risk aversion times, the instantaneous covariance between the growth rates of consumption and dividends ($\gamma\,\mathrm{Cov}_t(\Delta c, \Delta d)$), the LRR and the SRR. Returns are in percent.

**Table XI**. Predictive Regression Coefficients

| | Market Excess Returns | | | V-G Returns |
|---|---|---|---|---|
| | C | G | V | C |
| Model | -216.417 | 11.597 | -3.213 | 167.466 |
| Data | -202.475 | 10.246 | -6.265 | 81.677 |
| SE | 56.081 | 4.934 | 3.067 | 23.986 |

This table reports the the coefficients of predictive regressions in model (Model) and data (Data), and standard errors of estimates in data (SE). In the first three columns, we predict 4-quarter horizon future market-excess returns using SRR in consumption (C), dividends of growth stocks (G) and value stocks (V). In the last column, SRR in consumption (C) is used to predict 12-quarter horizon value-minus-growth returns.

**Table XII**. Forecast Future Excess Returns with SRRs (Adjusted for Repurchases)

| | Market Excess Returns | | | Value-Minus-Growth Returns | | |
|---|---|---|---|---|---|---|
| | 4Q | 8Q | 12Q | 4Q | 8Q | 12Q |
| C | $-194.006^{***}$ | $-93.434^{**}$ | $-37.034$ | $79.491^{***}$ | $86.510^{***}$ | $81.961^{***}$ |
| | (55.536) | (41.937) | (31.696) | (26.781) | (25.305) | (21.195) |
| G | 0.180 | 0.340 | $0.477^{**}$ | $0.405^{**}$ | $-0.134$ | $-0.265^{*}$ |
| | (0.453) | (0.316) | (0.207) | (0.206) | (0.205) | (0.143) |
| V | $-1.844^{*}$ | $-1.547^{***}$ | 0.009 | 0.654 | $0.903^{**}$ | $0.501^{*}$ |
| | (1.003) | (0.590) | (0.450) | (0.631) | (0.383) | (0.302) |
| Constant | $10.959^{***}$ | $7.760^{***}$ | $5.457^{***}$ | 0.770 | 0.688 | 0.985 |
| | (2.248) | (2.145) | (2.072) | (1.598) | (1.405) | (1.228) |
| Adjusted $R^2$ | 0.133 | 0.090 | 0.010 | 0.057 | 0.133 | 0.161 |

*Note:* $^{*}$p<0.1; $^{**}$p<0.05; $^{***}$p<0.01

This table reports the results of predictive regressions. Independent variables include $\text{SRR}^c_{t-1,t}$ (C), $\text{SRR}^g_{t-1,t}$ (G), $\text{SRR}^v_{t-1,t}$ (V). The first (last) three columns report results from regression forecasting the future 4, 8 and 12 quarters market excess returns (value-minus-growth returns). Newey-West standard errors with lag 8 are shown in parentheses, based on which the significance level is determined. The variances and covariances are represented in squared percent (%%), and returns are represented in percent (%).

**Table XIII**. Annualized Moments and Errors

| | Log-Lin | Real | Error | | Log-Lin | Real | Error |
|---|---|---|---|---|---|---|---|
| $\mathbb{E}(r_f)$ | 1.1353 | 1.1456 | 0.90% | $\mathbb{E}(r_e^g)$ | 4.7788 | 4.7474 | 0.66% |
| $\sigma(r_f)$ | 1.4091 | 1.3984 | 0.77% | $\sigma(r_e^g)$ | 7.5050 | 7.5004 | 0.06% |
| $\mathbb{E}(r_e^m)$ | 4.8350 | 4.8216 | 0.28% | $\mathbb{E}(p^g - d^g)$ | 3.9680 | 3.9671 | 0.02% |
| $\sigma(r_e^m)$ | 10.2310 | 10.2492 | 0.18% | $\mathbb{E}(r_e^v)$ | 7.1432 | 7.1916 | 0.67% |
| $\mathbb{E}(p^m - d^m)$ | 3.6494 | 3.6487 | 0.02% | $\sigma(r_e^v)$ | 20.2725 | 20.3363 | 0.31% |
| | | | | $\mathbb{E}(p^v - d^v)$ | 3.3703 | 3.3701 | 0.01% |

This table shows the means and standard deviations of the annualized risk-free rate, the market excess returns, the excess returns of growth (value) stocks. The table also reports the means of logarithms of price-dividend ratios. The relative errors in moments are determined through dividing the absolute difference between the projection method (Real) and the log-linearization (Log-Lin) moments by the real moment. The model moments are calculated via Monte-Carlo method by taking the average of a simulated sample of 2,400,000 monthly data.

**Table XIV**. Forecast Future Returns with SRIRs

| | Market Excess Returns | | | Value-Minus-Growth Returns | | |
|---|---|---|---|---|---|---|
| | 4Q | 8Q | 12Q | 4Q | 8Q | 12Q |
| Ind | 1.097 | 0.702 | 0.671 | 2.407 | 2.102 | 2.009** |
| | (1.412) | (1.328) | (1.112) | (1.703) | (1.283) | (0.812) |
| G | −1.127 | 0.348 | 0.638* | −0.495 | −0.431 | −0.559 |
| | (1.158) | (0.606) | (0.329) | (0.845) | (0.717) | (0.698) |
| V | −0.657** | −0.099 | −0.029 | 0.347** | 0.455*** | 0.303*** |
| | (0.307) | (0.174) | (0.123) | (0.155) | (0.109) | (0.115) |
| Constant | 4.171* | 4.059* | 3.995** | 2.790* | 3.072** | 3.150** |
| | (2.438) | (2.112) | (1.741) | (1.679) | (1.481) | (1.316) |
| Adjusted $R^2$ | 0.013 | −0.004 | 0.006 | 0.015 | 0.052 | 0.061 |

*Note:* *p<0.1; **p<0.05; ***p<0.01

This table reports the results of predictive regressions. Independent variables include $\widehat{\text{SRIR}}^c_{t-1,t}$ (Ind), $\widehat{\text{SRIR}}^g_{t-1,t}$ (G), $\widehat{\text{SRIR}}^v_{t-1,t}$ (V). The first (last) three columns report results from regression forecasting future 4, 8 and 12 quarters market excess returns (value-minus-growth returns). Newey-West standard errors with eight lag are shown in parentheses, based on which significance level is determined. SRIRs are represented in squared percent (%%), and returns are represented in percent (%).

**Table XV**. Theoretical Moments

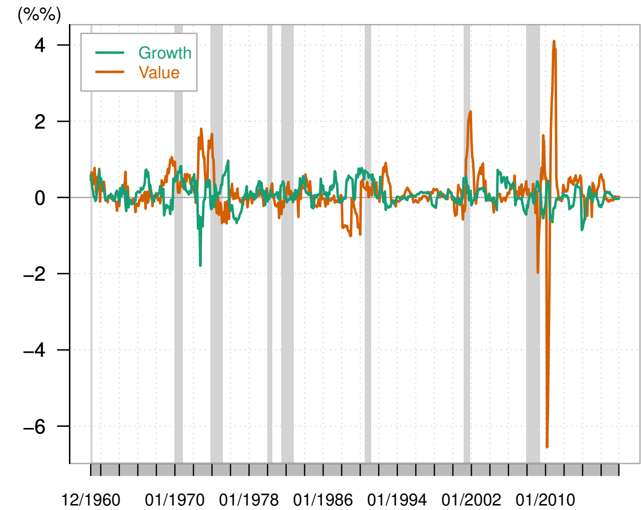
$$
\begin{aligned}
\mathbb{E}[\Delta c_t] &= \mu_c \\
\sigma(\Delta c_t) &= \sqrt{\delta_c'\Sigma(\infty)\delta_c + \operatorname{Tr}(\chi_c\Sigma(\infty)) + \bar{\sigma}_c^2 + \frac{\alpha - 1 + e^{-\alpha}}{\alpha^3}\delta_x'\Sigma(\infty)\delta_x} \\
AC1(\Delta c_t) &= \frac{\frac{1-2e^{-\alpha}+e^{-2\alpha}}{2\alpha^3}\delta_x'\Sigma(\infty)\delta_x}{\frac{\alpha-1+e^{-\alpha}}{\alpha^3}\delta_x'\Sigma(\infty)\delta_x + \delta_c'\Sigma(\infty)\delta_c + \operatorname{Tr}(\chi_c\Sigma(\infty)) + \bar{\sigma}_c^2} \\
\mathbb{E}[\Delta d_t^i] &= \mu_i \\
\sigma(\Delta d_t^i) &= \sqrt{\delta_i'\Sigma(\infty)\delta_i + \frac{\alpha - 1 + e^{-\alpha}}{\alpha^3}\delta_x'\Sigma(\infty)\delta_x\phi_i^2 + \sigma_i^2} \\
AC1(\Delta d_t^i) &= \frac{\frac{1-2e^{-\alpha}+e^{-2\alpha}}{2\alpha^3}\delta_x'\Sigma(\infty)\delta_x\phi_i^2}{\delta_i'\Sigma(\infty)\delta_i + \frac{\alpha-1+e^{-\alpha}}{\alpha^3}\delta_x'\Sigma(\infty)\delta_x\phi_i^2 + \sigma_i^2} \\
\operatorname{Cov}(\Delta d_t^i, \Delta c_t) &= \frac{\alpha - 1 + e^{-\alpha}}{\alpha^3}\phi_i\delta_x'\Sigma(\infty)\delta_x + \delta_i'\Sigma(\infty)\delta_c \\
\operatorname{Cov}(\Delta d_t^i, \Delta d_t^j) &= \frac{\alpha - 1 + e^{-\alpha}}{\alpha^3}\phi_i\phi_j\delta_x'\Sigma(\infty)\delta_x + \delta_i'\Sigma(\infty)\delta_j \\
\mathbb{E}[r_f] &= r_0 + \operatorname{Tr}(r_\Sigma\Sigma(\infty)) \\
\sigma(r_f) &= \sqrt{\frac{\alpha - 1 + e^{-\alpha}}{\alpha^3}r_x^2\delta_x'\Sigma(\infty)\delta_x + \operatorname{Var}(\operatorname{Tr}(r_\Sigma\int_t^{t+1}\Sigma_s ds))} \\
AC1(r_f) &= \frac{\frac{1-2e^{-\alpha}+e^{-2\alpha}}{2\alpha^3}r_x^2\delta_x'\Sigma(\infty)\delta_x + \operatorname{Cov}(\operatorname{Tr}(r_\Sigma\int_t^{t+1}\Sigma_s ds), \operatorname{Tr}(r_\Sigma\int_{t+1}^{t+2}\Sigma_u du))}{\frac{\alpha-1+e^{-\alpha}}{\alpha^3}r_x^2\delta_x'\Sigma(\infty)\delta_x + \operatorname{Var}(\operatorname{Tr}(r_\Sigma\int_t^{t+1}\Sigma_s ds))} \\
\mathbb{E}[r_e^i] &= \gamma\delta_c'\Sigma(\infty)\delta_i + (\frac{1-\psi\gamma}{1-\gamma})A_1A_{1i}\delta_x'\Sigma(\infty)\delta_x + 4(\frac{1-\psi\gamma}{1-\gamma})\operatorname{Tr}(QA_{2i}\Sigma(\infty)A_2Q) \\
\sigma(r_e^i) &= \sqrt{A_{1i}^2\delta_x'\Sigma(\infty)\delta_x + \delta_i'\Sigma(\infty)\delta_i + 4\operatorname{Tr}(QA_{2i}\Sigma(\infty)A_{2i}Q) + \operatorname{Var}\left(\operatorname{Tr}(\lambda_e^i\int_t^{t+1}\Sigma_s ds)\right) + \sigma_i^2} \\
AC1(r_e^i) &= \frac{\operatorname{Cov}\left(\operatorname{Tr}(\lambda_e^i\int_t^{t+1}\Sigma_s ds), \operatorname{Tr}(\lambda_e^i\int_{t-1}^{t}\Sigma_u du)\right)}{\sigma(r_e^i)^2}
\end{aligned}
$$

This table shows the expressions of model-implied moments. For notational simplicity, we denote the coefficient on $\Sigma_t$ for excess returns of portfolio $i$ by

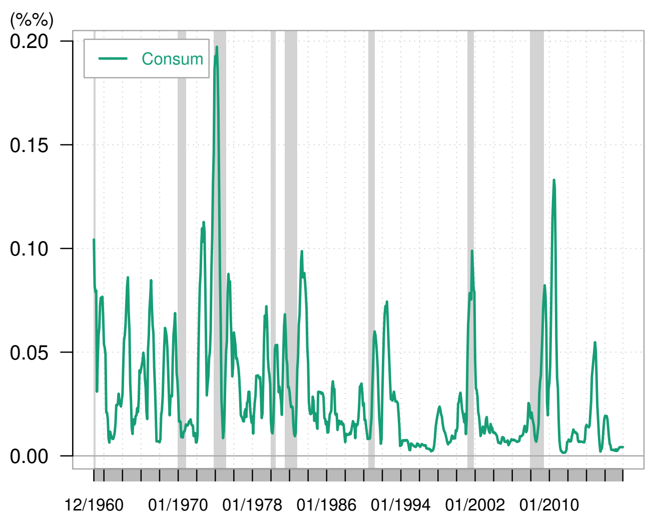
$$\lambda_e^i := \gamma\delta_i\delta_c' + (\frac{1-\psi\gamma}{1-\gamma})A_1A_{1i}\delta_x\delta_x' + 4(\frac{1-\psi\gamma}{1-\gamma})A_2QQA_{2i}$$

# Figures

**Figure 1.** SRRs



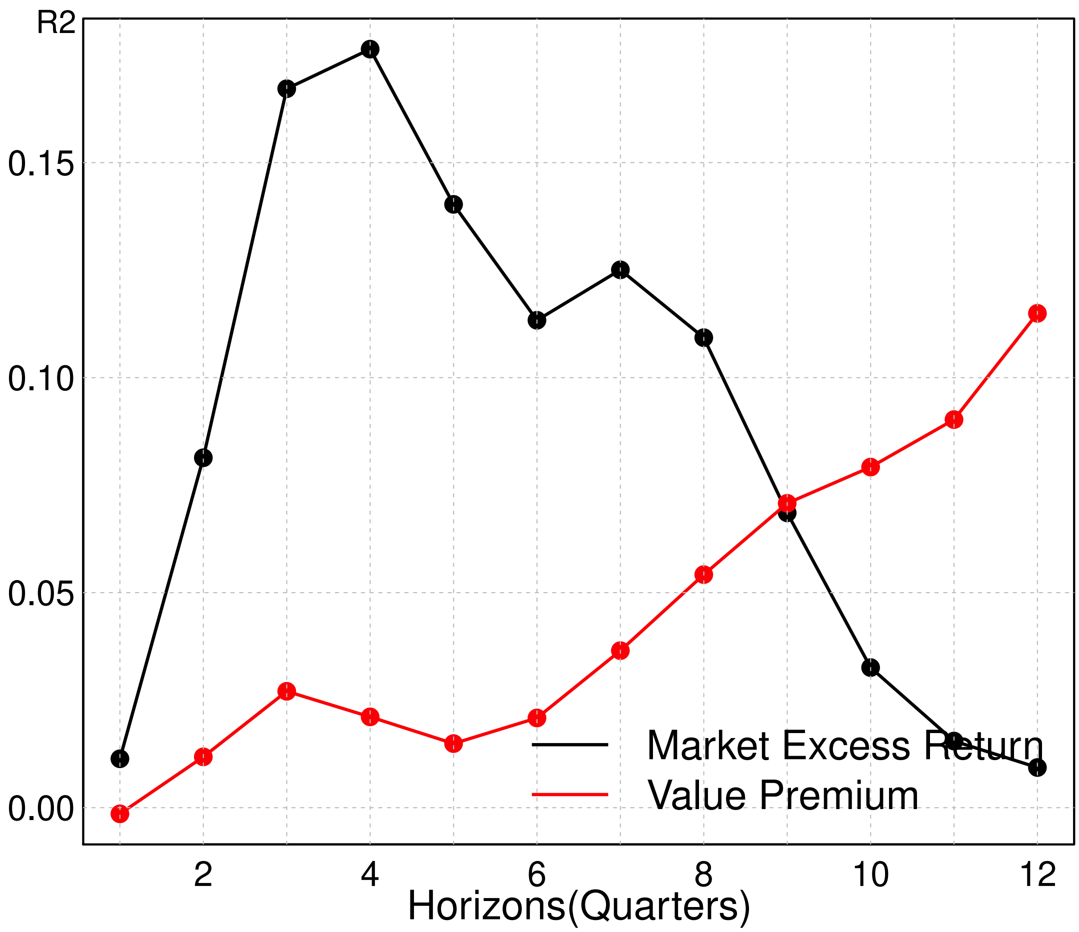

Panel A




Panel B

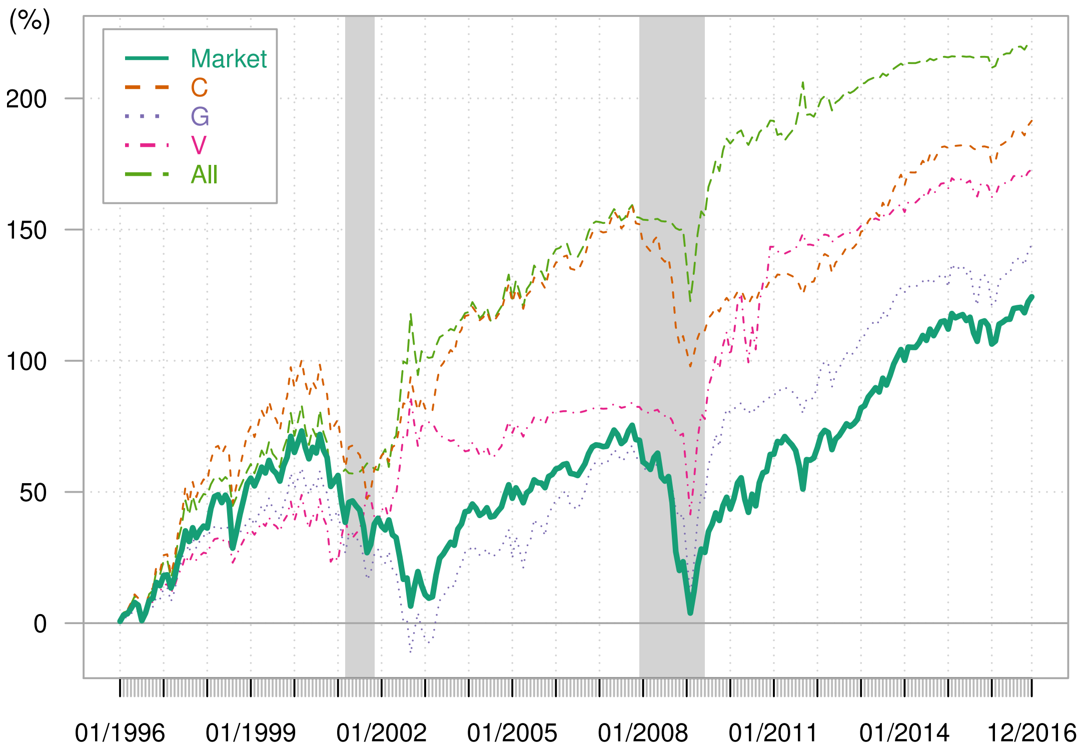
In Panel A, we plot the monthly observations of SRRs in value and growth stocks. In Panel B, we plot the monthly observations of SRRs in consumption. The shaded areas are NBER recorded recession periods. To construct value and growth portfolios, we use the upper and lower 30 percentiles of the value-weighted portfolios formed on book-to-market of individual stocks. Our sample ranges from 1959 to 2017. In $k$-th month in year $t$, we calculate the SRRs according to Equations (7) and (8). SRRs are represented in squared percent (%%).

**Figure 2.** Adjusted $R^2$ of Predictive Regressions Over Different Horizons



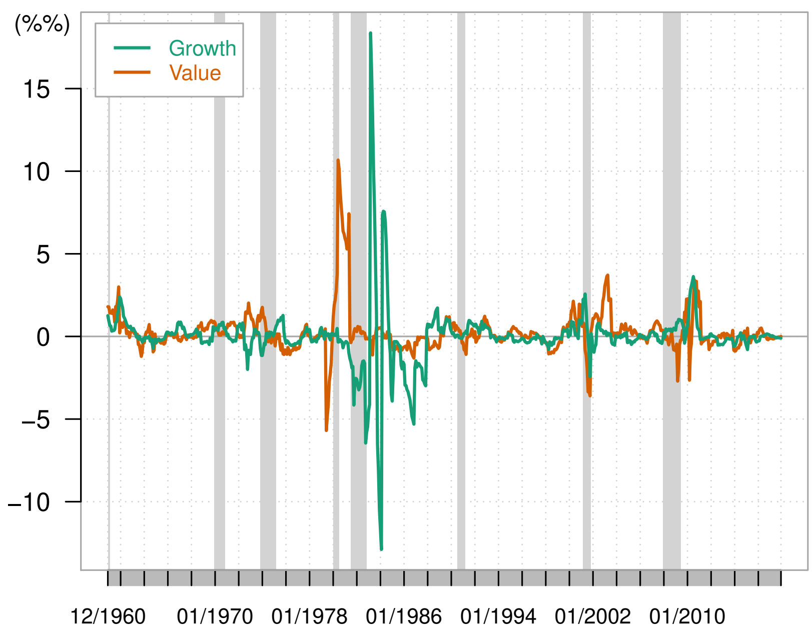

This figure plots the adjusted $R^2$s (in percent) of the predictive regression of future returns on SRRs specified by Equation (9). The dependent variables are the future market excess returns or value-minus-growth returns at future $Q$-quarter horizons. The independent variables are the SRRs in consumption, value and growth stocks. The black line shows the adjusted $R^2$ for predicting the future market excess returns, and the red line the adjusted $R^2$ for predicting the future value-minus-growth returns.

**Figure 3.** Cumulative Excess Returns



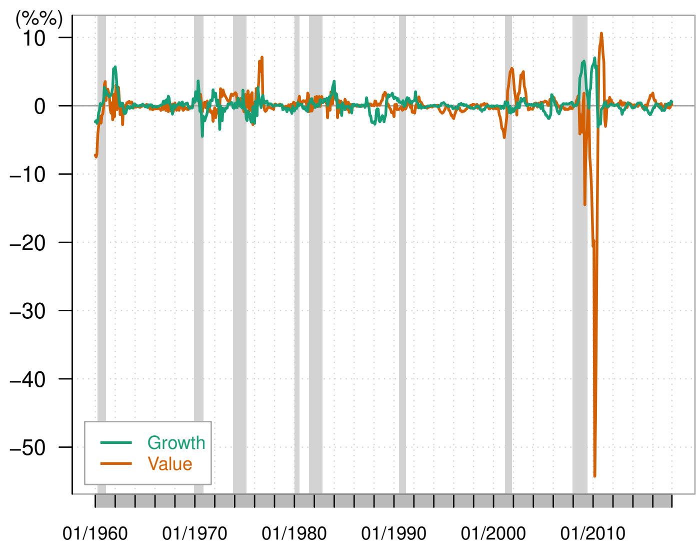

We plot cumulative excess returns using different investment strategies starting at the end of 1995. In each such strategy, we multiply the position in the market excess returns by the predictions from the out-of-sample regressions using: $\mathrm{SRR}^{c}_{t-1,t}$ (C), $\mathrm{SRR}^{g}_{t-1,t}$ (G), $\mathrm{SRR}^{v}_{t-1,t}$ (V) or all of the three SRRs (All). As a benchmark, we plot the cumulative market excess returns. We rescale the weights on the market excess returns ex post to ensure that all of the portfolios have the same volatility in returns. The out-of-sample predictive regressions estimate coefficients on a rolling basis from the past 35 years of data, and the weight on the market excess returns is updated at the beginning of each year.

**Figure 4.** SRRs in Value and Growth Stocks (Adjusted for Repurchases)

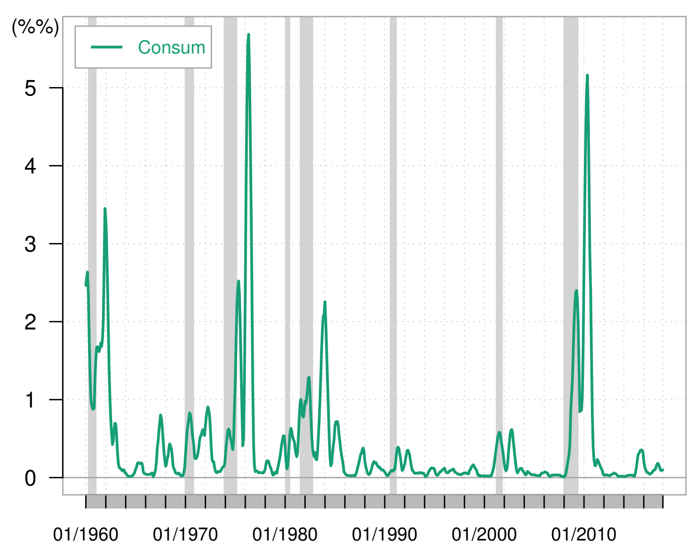



This figure displays the monthly observations of SRRs (adjusted for repurchases) in value and growth stocks. The shaded areas are NBER recorded recession periods. For the portfolios of value and growth stocks, we use the upper and lower 30% percentiles of the value-weighted portfolios formed on book-to-market. Our sample ranges from 1959–2017.

**Figure 5.** SRIRs



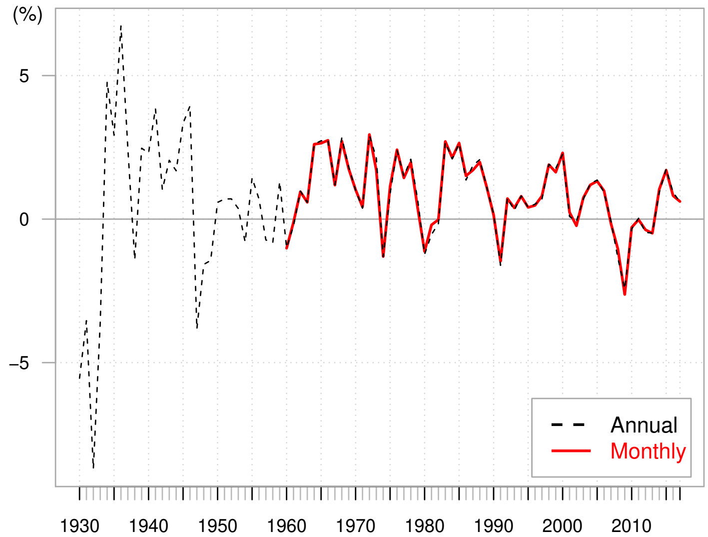

Panel A



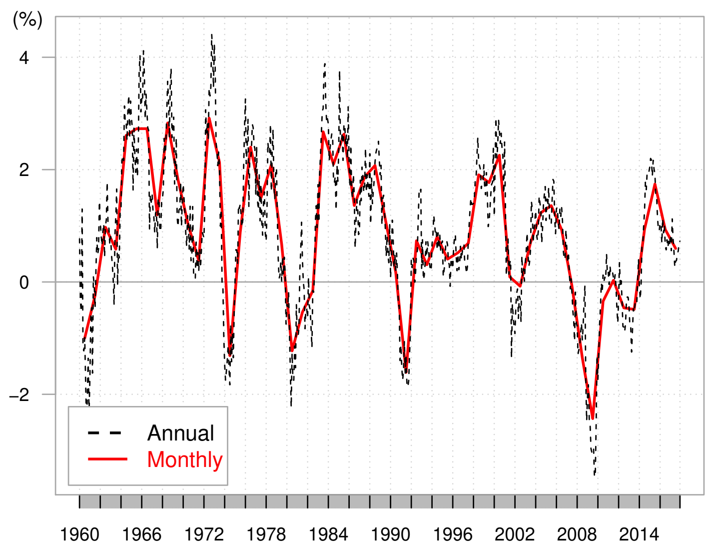

Panel B

In Panel A, we plot the monthly observations of SRIRs in the value and growth portfolios. In Panel B, we plot the monthly observations of SRIRs in industrial production index growth. The shaded areas are NBER recorded recession periods. For the value and growth portfolios, we use the upper and lower 30 percentiles of the value-weighted portfolios formed on book-to-market. Our sample ranges from 1959–2017. We estimate the SRIRs similar to Equations (7) and (8), with consumption replaced by industrial production index. SRIRs are represented in squared percent (%%).

**Figure 6.** Annual Consumption Growth Constructed from Annual and Monthly Aggregated Consumptions




Panel A




Panel B

Panel A plots the log annual growth rates of annually aggregated consumptions calculated as $\log \frac{C_{t,t+1}}{C_{t-1,t}}$ (dashed line) and that calculated from summing up the log monthly growth rates within a year of annually aggregated consumptions $\sum_{k=1}^{12} \log \frac{C_{t+(k-1)/12-1,t+(k-1)/12}}{C_{t+k/12-1,t+k/12}}$ (solid line). Panel B plots the annualized ($\times 12$) log monthly growth rate of annually aggregated consumptions calculated as $\log \frac{C_{t+(k-1)/12-1,t+(k-1)/12}}{C_{t+k/12-1,t+k/12}}$ (dash line) together with the log annual growth rates of annually aggregated consumptions $\log \frac{C_{t,t+1}}{C_{t-1,t}}$ (solid line). All growth rates are in percent.